PURE MATHEMATICS

MULTIPLE ANTANAIRESIS

BY

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1. Recently VIGGO BRUN (Norway) referred to the manner in which Aristotle approached the ratio of two incommensurable quantities. It was called *antanáiresis*. That means alternate taking away, alternate cancelling. Here is an example of the procedure. We take the quantities 1200 and 701,96 ... (being the musical intervals of the octave and the fifth measured in cents). The alternate subtractions are here shown.

	1200,000...	701,96...
1 × 701,96 =	701,96	
1 ×	498,04	= 498,04
2 × 203,92 =	407,84	203,92
2 ×	90,20	= 180,40
3 × 23,52 =	70,56	23,52
1 ×	19,64	= 19,64
5 × 3,88 =	19,40	3,88
	0,24	

We shall proceed no farther. For doing that we ought to know more digits in the number 701,96..... Operating on an irrational quantity, the procedure will never stop. The outcome of the antanáiresis is the infinite series of aliquot numbers

$$1,\ 1,\ 2,\ 2,\ 3,\ 1,\ 5,\ \ldots$$

which define the incommensurable ratio. If the infinite series is broken off, by arbitrarily equating to zero one of the remainders of the alternate subtractions, one gets an approximation of the incommensurable ratio in integer numbers. Cutting off after the first 1, the resulting ratio 1 : 1 is not yet a useful approximation. Cutting short after 1, 1, one gets 2 : 1. Successively one finds

1, 1, 2	yields a ratio	5 : 3	= 1200 : 720
1, 1, 2, 2	,, ,, ,,	12 : 7	= 1200 : 700
1, 1, 2, 2, 3	,, ,, ,,	41 : 24	= 1200 : 702,44
1, 1, 2 2, 3, 1	,, ,, ,,	53 : 31	= 1200 : 701,89
1, 1, 2, 2, 3, 1, 5	,, ,, ,,	306 : 179	= 1200 : 701,96

This antanáiresis comes down to the procedure of continued fractions.

2. The antanáiresis may be presented in *geometrical form*, taking the incommensurable numbers to represent the *components of a vector*, each being multiplied into a unit vector, $\mathbf{e}$, and $\boldsymbol{f}$. Thus

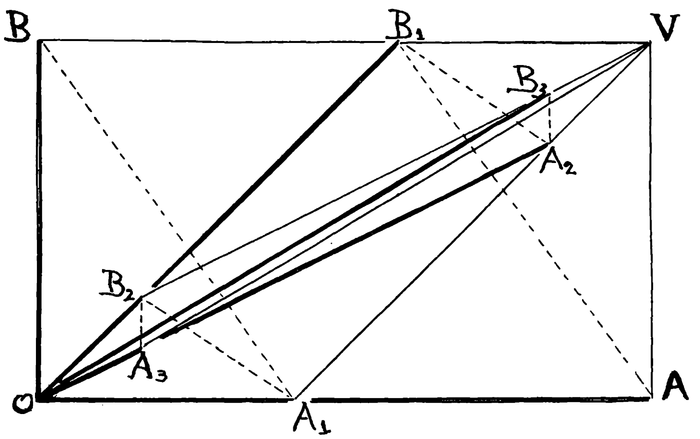
$$\boldsymbol{V} = A\mathbf{e} + B\boldsymbol{f}, \qquad (A > B).$$

The antanáiresis transforms the components, by writing

$$\boldsymbol{V} = (A - B)\mathbf{e} + B(\mathbf{e} + \boldsymbol{f}).$$

In our example we find a succession of transformations

$$\begin{aligned}
\boldsymbol{V} &= 1200\,\mathbf{e} + 701{,}96\boldsymbol{f} \\
&= 498\,\mathbf{e} + 702\,(\mathbf{e} + \boldsymbol{f}) \\
&= 498\,(2\mathbf{e} + \boldsymbol{f}) + 204\,(\mathbf{e} + \boldsymbol{f}) \\
&= 90\,(2\mathbf{e} + \boldsymbol{f}) + 203{,}92(5\mathbf{e} + 3\boldsymbol{f}) \\
&= 90{,}20(12\mathbf{e} + 7\boldsymbol{f}) + 23{,}52(5\mathbf{e} + 3\boldsymbol{f}) \\
&= 19{,}64(12\mathbf{e} + 7\boldsymbol{f}) + 23{,}52(41\mathbf{e} + 24\boldsymbol{f}) \\
&= 19{,}64(53\mathbf{e} + 31\boldsymbol{f}) + 3{,}88(41\mathbf{e} + 24\boldsymbol{f}) \\
&= 0{,}24(53\mathbf{e} + 31\boldsymbol{f}) + 3{,}88(306\mathbf{e} + 179\boldsymbol{f}).
\end{aligned}$$



Antanairesis with vectors

$$\begin{array}{lll}
OA = 12\,\mathbf{e} & OB = 7\,\boldsymbol{f}, & \\
OA_1 = 5\,\mathbf{e}, & OB_1 = 7\,(\mathbf{e} + \boldsymbol{f}), & A_1A = BB_1 \\
OA_2 = 5\,(2\,\mathbf{e} + \boldsymbol{f}), & OB_2 = 2\,(\mathbf{e} + \boldsymbol{f}), & B_2B_1 = A_1A_2 \\
OA_3 = 1\,(2\,\mathbf{e} + \boldsymbol{f}), & OB_3 = 2\,(5\,\mathbf{e} + 3\,\boldsymbol{f}), & A_3A_2 = B_2B_3
\end{array}$$

It is evident that the irrational coefficients get smaller and smaller, and the rational vectors within brackets grow larger and larger, approaching parallelism to the irrational vector $\boldsymbol{V}$.

3. The reason for this *increasing parallelism* is seen as follows. Comparing

$$\boldsymbol{V} = a\boldsymbol{u} + b\boldsymbol{w} \qquad (a > b)$$

and

$$\boldsymbol{V} = (a - b)\boldsymbol{u} + b(\boldsymbol{u} + \boldsymbol{w})$$

the areas of the parallelograms based on the two component vectors show to be

$$a \cdot b \cdot \mathbf{u} \times \mathbf{w} \text{ and } (ab - b^2) \cdot \mathbf{u} \times (\mathbf{u} + \mathbf{w}) = (ab - b^2) \cdot \mathbf{u} \times \mathbf{w}.$$

The area has decreased, the diagonal is conserved. Therefore the angle between the diagonal and one of the sides must have diminished.

In the antanáiresis, the side vectors approaching the diagonal vector, their commensurable direction coefficients provide approximations to the ratios of the incommensurable direction coefficients of the diagonal vector.

4. VIGGO BRUN, from whom I borrow the word antanáiresis, discovered the algorithm for extending antanáiresis to *three*, and more *irrational quantities*. I translate his algorithm into vector form, representing the irrational quantities A, B, C (with $A > B > C$), with the aid of unit vectors $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$, by an irrational vector

$$\mathbf{V} = A\mathbf{e} + B\mathbf{f} + C\mathbf{g}.$$

The first step leads to $\mathbf{V} = (A - B)\mathbf{e} + B(\mathbf{e} + \mathbf{f}) + C\mathbf{g}$. Picking out the largest and the second largest of the numbers $(A - B)$, B, C, we make the second similar step with the component vectors involved. The result is a series of transformations

$$\mathbf{V} = a_n \mathbf{u}_n + b_n \mathbf{v}_n + c_n \mathbf{w}_n, \text{ (with } b_n > c_n > a_n, \text{ say)}$$

$$= a_n \mathbf{u}_n + (b_n - c_n)\mathbf{v}_n + c_n(\mathbf{v}_n + \mathbf{w}_n).$$

Taking the volume of the parallelepipedum formed by the new vectors we find

$$a_n(b_n - c_n)c_n \cdot \mathbf{u}_n \times \mathbf{v}_n \times (\mathbf{v}_n + \mathbf{w}_n) = a_n(b_n c_n - c_n^2)\, \mathbf{u}_n \times \mathbf{v}_n \times \mathbf{w}_n.$$

It has diminished and will continue to diminish at each step. The diagonal vector remaining constant the edge vectors of the parallelepipedum will approach the direction of the diagonal.

5. I shall *apply the antanáiresis* to *three numbers* representing musical intervals, *vid.* the octave, $O = 1200$, the perfect fifth, $Q = 701{,}96$ and the perfect major third, $T = 386{,}31$. One starts with the vector

$$\mathbf{V} = O\mathbf{e} + Q\mathbf{f} + T\mathbf{g}$$

$$= 1200\mathbf{e} + 702\mathbf{f} + 386\mathbf{g}.$$

Carrying out the operations according to the given rules, the tenth line after this reads

$$\mathbf{V} = 14(19\mathbf{e} + 11\mathbf{f} + 6\mathbf{g}) + 22(34\mathbf{e} + 20\mathbf{f} + 11\mathbf{g}) + 6(31\mathbf{e} + 18\mathbf{f} + 10\mathbf{g}).$$

Taking away 14 from 22, the next line reads

$$\mathbf{V}=14(53\mathbf{e}+31\mathbf{f}+17\mathbf{g})+8(34\mathbf{e}+20\mathbf{f}+11\mathbf{g})+6(31\mathbf{e}+18\mathbf{f}+10\mathbf{g}).$$

Within brackets we have rational vectors nearly parallel to $\mathbf{V}$. Calling the approximating proportional substitutes o, q, t, we notice, besides the familiar division of the octave in semitones

$o:q:t=12:7:4$ of the keyboards of organ and pianoforte,

$o:q:t=19:11:6$, the division in *thirds of a tone,* advocated and used by WOOLHOUSE and by YASSER,

$o:q:t=31:18:10$, the division in *dièses,* discovered by VICENTINO and calculated by HUYGENS,

$o:q:t=34:20:11$, sometimes supported by DROBISCH,

$o:q:t=53:31:17$, the division due to MERCATOR.

6. To apply the *antanáiresis* to *four numbers* we add the interval representing the *perfect seventh,* $S=968{,}83$.

The initial vector

$$\mathbf{V}=1200\mathbf{e}+968{,}83\mathbf{f}+701{,}96\mathbf{g}+386{,}31\mathbf{h}$$

in twelve antanairetical steps is transformed into

$$\begin{aligned}\mathbf{V}= {} & 5{,}37(35\mathbf{e}+28\mathbf{f}+20\mathbf{g}+11\mathbf{h})+\\ & +13{,}82(53\mathbf{e}+43\mathbf{f}+31\mathbf{g}+17\mathbf{h})+\\ & +13{,}08(15\mathbf{e}+12\mathbf{f}+9\mathbf{g}+5\mathbf{h})+\\ & +2{,}69(31\mathbf{e}+25\mathbf{f}+18\mathbf{g}+10\mathbf{h}).\end{aligned}$$

We already met the approximating vectors with $31\mathbf{e}$ and $53\mathbf{e}$. The approximation $15:12:9$ for $o:s:q$ is inferior, because the differences between o, s and q ought not be equal, $(O-S)<(S-Q)$.

7. An example of *five numbers* is afforded by adding the *perfect eleventh,* $U=551{,}32$ .... The initial vector is

$$\mathbf{V}=1200\mathbf{e}+968{,}83\mathbf{f}+701{,}96\mathbf{g}+551{,}32\mathbf{h}+386{,}31\mathbf{k}.$$

After sixteen antanairetical operations one finds

$$\begin{aligned}\mathbf{V}= {} & 12{,}98(72\mathbf{e}+58\mathbf{f}+42\mathbf{g}+33\mathbf{h}+23\mathbf{k})+\\ & +4{,}40(17\mathbf{e}+14\mathbf{f}+10\mathbf{g}+8\mathbf{h}+6\mathbf{k})+\\ & +0{,}84(63\mathbf{e}+51\mathbf{f}+37\mathbf{g}+29\mathbf{h}+20\mathbf{k})+\\ & +0{,}55(41\mathbf{e}+33\mathbf{f}+24\mathbf{g}+19\mathbf{h}+13\mathbf{k})+\\ & +3{,}11(37\mathbf{e}+30\mathbf{f}+22\mathbf{g}+17\mathbf{h}+12\mathbf{k}).\end{aligned}$$

It must be noted that nowhere in the preceding lines terms like $12\mathbf{e}$, $19\mathbf{e}$, $31\mathbf{e}$, $34\mathbf{e}$, $53\mathbf{e}$ have appeared.

For sets of numbers approximating the ratios between O, S, Q, U and T we here meet

$$o:s:q:u:t=17:14:10:\ 8:\ 6$$
$$o:s:q:u:t=37:30:22:17:12$$
$$o:s:q:u:t=41:33:24:19:13$$
$$o:s:q:u:t=63:51:37:29:20$$
$$o:s:q:u:t=72:58:42:33:23.$$

The first set is much to coarse. It puts $3t>O$, where $3T<O$. The second set is defective, because it puts $q-u=u-t$, whereas $(Q-U)<(U-T)$. The third set has been proposed by JANKO.

8. In order to estimate the *degree of accuracy* of the approximation it is obvious to look for the angle which the approximating rational vector makes with the irrational vector **V**. One needs the direction coefficients of both. In the last example, that of the five numbers, the direction coefficients for the irrational vector are

$$O:S:Q:U:T = 0{,}6582 : 0{,}5313 : 0{,}3850 : 0{,}3022 : 0{,}2118.$$

For the approximating rational vectors one finds respectively

$$41:33:24:19:13 = 0{,}6585 : 0{,}5300 : 0{,}3855 : 0{,}3052 : 0{,}2088$$
$$63:51:37:29:20 = 0{,}6575 : 0{,}5323 : 0{,}3862 : 0{,}3027 : 0{,}2087$$
$$72:58:42:33:23 = 0{,}6593 : 0{,}5311 : 0{,}3846 : 0{,}3022 : 0{,}2105.$$

The sum of squares of the deviations amounts to

$$20\cdot 10^{-6},\quad 12\cdot 10^{-6},\quad 2\cdot 10^{-6}$$

respectively. There is a moderate gain in passing from 41 to 63. A considerable gain is secured in passing from 63 to 72.

It might be interesting to see how the accuracy compares if one tries to use an approximation with 53, the best in the preceding example of four irrationals.

The direction coefficients are

$$53:43:31:24:17 = 0{,}6582 : 0{,}5340 : 0{,}3850 : 0{,}2981 : 0{,}2111$$

and the sum of squares of the deviations is $25\cdot 10^{-6}$, comparing unfavourably with the deviations found above.

9. In the case of four intervals the comparison of the approximations runs as follows. For the irrational vector the direction coefficients are

$$O:S:Q:T=0{,}6904 : 0{,}5575 : 0{,}4039 : 0{,}2221.$$

For the approximating rational vectors they are

$$31:25:18:10 = 0{,}6915 : 0{,}5576 : 0{,}4015 : 0{,}2230$$
$$35:28:20:11 = 0{,}6958 : 0{,}5567 : 0{,}3976 : 0{,}2187$$
$$53:43:31:17 = 0{,}6895 : 0{,}5594 : 0{,}4033 : 0{,}2212.$$

The sum of squares of the deviation amounts to

$$8 \cdot 10^{-6}, \quad 80 \cdot 10^{-6}, \quad 6 \cdot 10^{-6}$$

respectively. The inferiority of the 35-approximation is very evident. The gain in passing from 31 to 53, though appreciable, is not overwhelming.

10. By his antanairetical method VIGGO BRUN has enabled us to answer questions which are important to musicologues and musicians who want to know where to look for improvement of the equal temperament of twelve semitones, in order to deal adequately with higher harmonies, such as the perfect seventh and the perfect eleventh.

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