Adriaan D. Fokker

New Music with 31 Notes

ORpheus

Translated by Leigh Gerdine

Volume 5 of the ORPHEUS Series of Monographs on Basic Questions in Music

edited by Martin Vogel

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Verlag für
systematische Musikwissenschaft
GmbH

Bonn - Bad Godesberg 1975

German edition 1966

TRANSLATOR'S PREFACE

Adriaan Daniel FOKKER was born on August 17, 1887 at Buitenzorg (now Bogor), Java, the son of the president of the factory at Batavia (now Jakarta) of the Netherlands Trading Society *(Nederlandsche Handelmaatschappij).* The family returned to the Netherlands in 1894. Adriaan Fokker took his doctorate in physics at the University of Leiden in 1913, working with the renowned Professor H. A. LORENTZ. He did advanced study with Albert EINSTEIN, Sir Ernest RUTHERFORD, and Sir William BRAGG. His career as a professor of physics began in the Technical University at Delft (1923-27). He then succeeded Professor LORENTZ as the head of the Physics Department of the Teyler Foundation at Haarlem and Teyler Professor of Physics in the University of Leiden.

During the German occupation of the Netherlands, Fokker was struck by the publication, in 1941, of the musicological studies of the 17th century Dutch physicist and mathematician, Christiaan HUYGENS, who calculated a *nouveau cycle harmonique* with the division of the octave into 31 dièses. The University of Leiden being closed during the occupation, and his regular research opportunities unavailable, Fokker began elaborating Huygens' discovery, with the help of Leonhard EULER's *Tentamen novae theoriae musicae* (1739).

Until the time of his death, September 24, 1972, Fokker continued to work with the 31-note scale which he had pioneered, to encourage musicians to work with it, and to sponsor concerts of music written in the system.

In this book, in the first part, the historic/anecdotal section, he sets forth the story of his interest in an improved tuning system with its vast potential, and then proceeds, in the second part, to a mathematical explanation and exploration of the elements of the new system. The reader who is primarily interested in the technical discussion may prefer to skip directly to Part Two, the Systematic Section.

It is too soon to sum up Fokker's achievements in any but the most superficial way. The future of the 31-note system – he called it *Tricesimoprimal* – has not been decided. It has scarcely been considered. Perhaps the reappearance of this book in English will give it some impetus.

Whatever the ultimate acceptance of the system, Fokker has accomplished a umber of highly desirable objectives:

1) he has established the physical basis for the existence of the minor triad and scale in the mirrored (reciprocal) versions of his interval relationships;
2) he has made his 7th an acceptable *harmonic* interval — where HINDEMITH was afraid to pursue it further because he was unsure where pursuing it might lead;
3) he has conceived — and had built — a keyboard which brilliantly solves the problems of 31-note music, and which eliminates some of the inadequacies of our present 12-note keyboard;
4) he has opened up the challenges of a rational new sound spectrum, which allows both a greater consonance and a greater dissonance, both a more authentic performance of earlier (pre-Bach) music and an expanded potential for the music of the future.

My own interest in new tuning systems goes back many years. Fascinated by Fokker's work, I went to see him in Haarlem in 1968. After meeting him, I arranged to issue a recording of *Tricesimoprimal Music*, including some very beautiful choral songs by Henk BADINGS. I still have a limited number of copies of that recording, available at Webster College. Probably some copies of it have found their way into music libraries here and there. Early in 1974, Webster College will have available a small, portable *archifoon*, an electronic keyboard instrument equipped to perform in the 31-note system. To my knowledge, this is the only such instrument in the United States.

When Professor Kendall STALLINGS of the Department of Music at Webster College had spent a summer working with Fokker, he suggested to me that this book ought to be available in English. I translated it in the spring of 1973, after getting the permission of Fokker's widow, T. FOKKER VAN DIJK.

There are two problems which I particularly need to call to the attention of the reader:

1) that I have used a system of translation of the German names for the new pitches which is my own, which is more cumbersome than the German, but which is relatively simple and perhaps adequate at this stage; it appears at Table I, with the German equivalents. I debated carrying over the German names, some of which are already familiar to many musicians, but in the end thought that an English equivalent was imperative;
2) the reader must understand that, in this new system of pitch names, the pitches themselves do NOT correspond to the pitches in our 12-note system which bear the same name. That cannot be sufficiently emphasized. They are approximations, of course. But they are not the same.

And although it may seem a side issue at this point to raise the question, I have long been privately convinced that we must re-record all pre-Bach music, utilizing an appropriate tuning system in order to give to earlier music a nearer approximation of its true beauty and vitality.

Finally, so brief a monograph leaves little space for acknowledgements. Dr. C. Kendall STALLINGS has helped me with careful proofreading of the text and has contributed a valuable table of intervals. Herr Albrecht SCHNEIDER made many valuable suggestions for improving and correcting the English language version. Dr. Paul PISK made many corrections of detail. Mrs. T. FOKKER-VAN DIJK, Fokker's widow, encouraged us, and kept the project alive. Martin VOGEL has been the guiding force in seeing this English language version through to completion. Anton DE BEER has contributed a section bringing the history of 31-tone music up-to-date, and has completed the table of works in the 31-tone system to include works not yet composed when the German original version was published. It would be derelict of me not to mention with awe the heroic job of transcription and preparation of the manuscript done by my very able secretary, Isobel GANTS.

Leigh Gerdine

Webster College
St. Louis, Missouri
November 1, 1973

THE PUBLISHER'S FOREWORD OF 1966

What direction music ultimately will take cannot be judged today. Predictions of this kind of course should not be the task of musicology. Musicology has to investigate musical material and by study make new material available. If we ask ourselves "What kind of music is appropriate to modern man?", we have already entered the field of problems which generated this book. It deals with pure thirds, pure sevenths and intervallic relations of higher prime numbers. How the ear comprehends intervals of prime numbers of higher orders (e. g. the 11th, 13th, etc.) has yet been little explored; of the pure third and pure seventh represented by the prime numbers five and seven, we know for certain that they are grasped by the ear, that they sound "good". A music which would take advantage of them would reach broad groups of listeners and would also fascinate professionals, to whom it would bring something new: a true enharmonic, the distinction of smaller and smallest pitch differences. New scales would stand at the command of the composer, new sounds, cadences, and methods of modulation, modulation into tonalities related through the seventh, for example.

At the time when keyboard instruments were tuned in mean-tone temperament, the aim was to reach pure thirds. Later, equal temperament, which represented a fairly good approximation to Pythagorean tuning, was generally introduced, while the thirds largely were given up and one fell back on tuning in fifths. With respect to the pure sevenths, they are not used today, although it has been well-known for 300 years that there is nothing from the aural side to prevent their acceptance and application in music. The group of intervals based on the seventh stands on the border between consonance and dissonance.

The present monograph acquaints us with a tonal system that has pure thirds and sevenths. With its 31 scale-degrees it has more than twice as many notes as are available today on a keyboard instrument. Musical laymen will suspect that one can no longer find his way through so large a number of notes. Our Dutch friends have found their way through it. The value of this monograph lies not the least in that it speaks not only of future possibilities, but, in its first, historic/anecdotal part, sets forth what has already been achieved. The composers

ANECDOTAL-HISTORICAL SECTION

The Hollandsche Maatschappij der Wetenschappen

When, in the middle of the eighteenth century, interest in the development o natural science was dominant everywhere and the period of the Enlightenment began, societies were formed in the various cities of the Republic of the Seven United Netherlands in which scholars and laity came together for the advancement of this enlightment. In Haarlem such a society was established, the *Hollandsche Maatschappij der Wetenschappen.* This society divided itself into two sub-groups, into the group of the *Directeuren,* to which the rich merchants and influential officials of the city belonged, and into the group of the *Leden* (mem bers), which numbered the most meritorious scientists. At their regular sessions the cause of science was served.

In the period of French sovereignty, King Louis Bonaparte sought to change the *Hollandsche Maatschappij* into an academy modeled after the Paris Academ of Science. The *Maatschappij* refused, however, to give up its double structure, the division into *Directeuren* and *Leden.* King Louis Bonaparte had to look for another solution.

Through the financing of its scientific interests, the *Hollandsche Maatschappı* pushed the development of science in the direction of physics. One of its most important contributions consists of the publication of the documents of Christi aan HUYGENS. The greater part of his estate lay in the University Library at Leiden. This important undertaking was begun in 1885 and completed in 1950 with the 22nd volume of the *Huygens Ausgabe.*

The Twentieth Volume of Huygens' Complete Works

In 1940 appeared the 20th Volume of Christiaan Huygens' works containing the musico-theoretical writings. The great physicist, astronomer, and mathematician was very gifted musically. As a young boy he was commissioned by his fathe Constantine Huygens, to teach sight singing to his older brother. As payment he received three shillings a week. In three weeks the older brother could sight-sing

. was then appropriate in the best society. The mentioned 20th volume con-
.ins notes and memoranda which Christiaan Huygens made on the reading of
.usico-theoretical works. One is astounded by the number of texts which he
udied.

Huygens refused, because of 12-tone tempering, to deny himself the beauty
: the triad, as it was performed in the middle of the 17th century and as it
as practiced by the artists on the lute. With the help of the calculus which
as invented at this time, he discovered a *Nouveau Cycle Harmonique* (New
armonic Circle) (he lived in Paris), which concluded not after 12 steps but,
›r the first time, only after 31. The smallest interval was the 31st part of an
:tave, the *diësis*, the fifth-tone. Huygens found that the 31-note octave offered
l of the advantages of equal temperament, especially that of unlimited modu-
tion through all of the keys, without requiring that the chords be mistuned
·udely. In fact, in the 31-note temperament the third and seventh are almost
ıre; only the fifths are about 5 cents, 5/100 of a half-step too low.

The New Harmonic Circle

Christiaan Huygens described it as follows: *We have here a system which con-
ins every step of the scale, every whole-step, half-step, and every* Diësis, *all the
nsonances and intervals from the highest to the lowest, alike in every octave.*
rom this arises the homogenity of the scale. Huygens continues: *In this system
e larger half-step has three-fifths of a whole-step, and the smaller half-step the
maining two-fifths. Then, it sets up a complete harmonic circle in which, by
sing or falling in fifths or any other interval, after a certain series, one reaches
e beginning step again.* Somewhat further on he says: *I would also here like
· note in favor of this new temperament that everywhere in it the interval of
e tritone is contained in the relationship of 7 to 5; in this numerical relation-
ip only 1/12th of a comma is missing . . . I now find that this interval 7 to 5,
one tests it carefully, has something harmonic – in any case I experience it
us with my ear – and one may count it among the consonances, no matter
hat their lordships, the composers, who counted it more often among the false
nal relationships, may say against it*[1].

With these words the characteristics of the 31-note scale are brought clearly
to focus: uniform homogeneity, and a closed cycle; differentiation between
rge and small half-steps (their difference is the diësis, the smallest intervallic
ıit of the 31-step tonal system); pure thirds and harmonic sevenths calculated
om every scale degree. How much interest was generated by this seventh not
ıly at the present time but in past centuries has been set forth in the works of
artin VOGEL in thorough and overwhelming detail.

Chr. Huygens, *Oeuvres complètes*, vol. XX, La Haye 1940, p. 160ff.

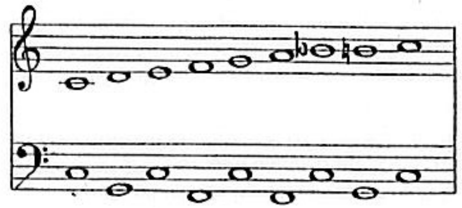
Taking a Position With Respect to the Seventh

The reading of Huygens' text encouraged me to ask musical friends and artists about their conception of the seventh. Zoltan SZEKELY, the leader of a Hungarian string quartet, explained to me, that the pure seventh is not used in music. When I once asked the famous violinist Carl FLESCH to play me the series of harmonics, he played the notes 4,5,6 and 8. When I called attention to the space between 6 and 8, he also played the 7th overtone. When I asked him why he had left it out and whether he sometimes used it, Flesch answered that he was no theorist. In the *Encyclopaedia Britannica*, Sir Donald Francis TOVEY wrote, in his article on "Harmony": *"The study of harmony has not yet found a place for so natural an appearance as the 7th note in the harmonic series."*

In RIEMANN's *Geschichte der Musiktheorie* (History of Music Theory, 2nd edition, Berlin 1920) I found a reference to Giuseppe TARTINI, who in his *Trattato di Musica* (Padua 1754) conceded to the pure seventh its appropriate place in the scale on C between *la (a)* and *si (b)*.

Scale:

Fundamental Bass:



T D T S T S T D T

Tartini reasoned that, in the major scale, a symmetrical bass leading could only be reached if a seventh lying over the tonic were inserted between the *la* over the sub-dominant and the *si* over the dominant. Through the symmetrical bass-leading Tartini easily got around the difficulties of the voice-leading which arise from the direct succession of sub-dominant and dominant.

When, at a lecture, I asked Willem PIJPER, the grand master of Dutch composers, to take a position on the diësis, he excused himself, saying that he could not be content with a superficial answer, that an authoritative, considered pronouncement would be expected of him. He thought it possible that a return to the diësis might bring a total change in the direction of music, about which he could however make no prediction. At about the turn of the year 1941/42, my colleague Dr. Balth. VAN DER POL and I gave lectures in the aula of the Teyler Museum at Haarlem on the diësis and the new chords with the seventh 7/4. For thoses lectures Van der Pol furnished very expressive acoustic experimen-

demonstrations. He was the head of the electronic section of the Laborato-
of the Philips *Gloeilampenfabrieken N. V.* in Eindhoven. Through his theo-
ical preparatory studies, the first radio telephone connection between the
therlands and Bandung, Java, was made possible. In his youth, he took com-
sition lessons with the Amsterdam composer Bernard ZWEERS. Van der Pol
led my attention to the musico-theoretical works of Leonhard EULER.
During the Haarlem lectures I became acquainted with Martinus LÜRSEN, the
ief instructor for music theory at the Royal Conservatory at the Hague. Lür-
n lived in Haarlem. He took a position, in friendly manner, open-minded to-
ird the seventh.

The Study of Older Theorists

During the German occupation I could not work much in my profession as a
ysicist. I turned to the study of musico-theoretical problems. I was able to
rrow various rare books from the music library of the *Vereeniging voor Ne-
rlandsche Muziekgeschiedenis* in Amsterdam, Gioseffo ZARLINO's the *Istitu-
ni Harmoniche* (1558), the *Dimostrationi Harmoniche* (1571) and the *Sop-
menti Musicali* (1588); Jean Philippe RAMEAU's *Traité de l'harmonie* (1722)
d the *Nouveau Système de Musique* (1726). Both Rameau's works are very
ng-windedly and vaguely conceived; one had better leave them unread. His
nération harmonique from the year 1737 makes a better impression; and fi-
lly the *Démonstration du principe de l'harmonie* which appeared in 1750 pre-
nts a clear representation of Rameau's teaching; the shortcomings of the early
oks are eliminated from it.
Of TARTINI, I studied the *Trattato di musica* from the year 1754. Tartini
ade use here of special alteration signs for the lowering of a fifth step ("semi-
it") and for a 3/5 step ("sesqui-flat"). Tartini's signs seemed to me very ne-
ssary and useful. I picked up Leonhard Euler's *Tentamen novae theoriae mu-
cae* of the year 1739.

The Classification of the Seventh

In Euler's essay on music theory, two of his central conceptions are empha-
zed particularly: his *Gradus Suavitatis* and his *Genera musica.*
Two notes (*a:b*) which have a rational frequency relationship to one another
wherein *a* and *b* are whole numbers, without common multiples other than 1)
ive not only a common fundamental (frequency equals 1), but also common
ertones. The lowest common overtone (frequency equals *a* times *b*) I call the
guidetone."[1] The distance between the fundamental and the guidetone (nume-

Fokker calls this the *Führungston*, which is technically *leading tone*. But because *leading*

rically *a* times *b*, called by Euler the "exponent") can be considered as an in-
verse measurement for the relationship of the notes and for the consonance of
the intervals. A lower *Gradus Suavitatis* corresponds to a larger exponent, thus
a greater tension.

The octave 1:2 has the exponent 2. The interval of the major sixth 3:5 has
the exponent 15. The pure seventh 4:7 has the exponent 28. The connecting
of the tonal pairs 3:5 and 4:7 creates the guidetone 3 x 5 x 4 x 7 = 420, hence
the exponent 420. From the factors of the exponent, Euler deduces his *Gradus
Suavitatis* in the course of complicated directions, which I will not discuss fur-
ther here. Summarizing Euler's system, we can here content ourselves by taking
the distance between fundamental und guidetones, that interval, as a measure
of the tension, and express this distance mathematically in terms of octaves.
The octave itself would thus have a tension of 1. The major sixth, with the ex-
ponent 15, is a large half-step (15:16) smaller than 4 octaves (16). If one reckons
that the half-step is approximately 1/10 of an octave, then the major sixth has
thus a tension of 4 - 1/10 = 3.9 octaves. The pure seventh has the exponent 28:
measured in octaves the tension number is 4.8 octaves. The pure four-note chord
3:4:5:7, with the exponent 420 arrives at an octave tension number of 8.7 octa-
ves. A further example: the $^6_4$ position of the common major triad 3:4:5, the
common major triad 4:5:6, and the common minor triad 10:12:15 all have the
same guidetone, 60. The exponent 60 is about one large half-step (1/10th of an
octave) smaller than 6 octaves (64). They all have thus the same tension num-
ber of 5.9 octaves. From another point of view Paul HINDEMITH, as is well
known, categorized intervals and chords according to their tension in his *Unter-
weisung im Tonsatz.* Euler has given it a numerical definition.

For the systematic construction of his *Genera musica,* Euler established a
note (for example, *c*) and transposed this note with the help of the basic inter-
vals (the fifth or major third). The note appears „multiplied" by this interval.
Thus the note *c* is multiplied through the interval of the fifth to the pair of
notes *c–g.* The pair could be "multiplied" with a further basic interval, for exam-
ple with the major third, whereby the tone pairs *c–g* and *e–b* arise. One might
now "multiply" a second time with the fifth, perhaps also downwards with the
fifth below, whereby the four-note series *c-e-g-b* and *f-a-c-e* result, with two
notes in common (*c* and *e*). The intervals made use of here, the fifth (3/2) twice
and the third (5/4) once, are decisive for the definition of the *Genus*, which, in
the case of our example, would be defined through the product of 3/2 x 3/2 x
5/4. Since the *Genus*, with its supply of notes would be valid for all octaves,
which implies multiplication with the factor 2 of the octave, the factors 2 are
superfluous for the definition of the *Genus.* We will represent the *Genus* by
placing the remaining factors in parentheses as (3.3.5).

tone has other connotations in musical terminology, I have chosen *guidetone* here in order to
avoid confusion. The usual German form for *leading tone* is *Leitton.* – Tr.

)

Obviously instead of multiplying with the fifth and the major third, one ight as well have multiplied with the harmonic seventh (7/4). This would be ıe way, as Euler showed us, to bring the seventh importantly into the con- ruction of the tonal system.

The simplest *Genera* arise from threefold multiplication. From the factors 3, and 7, the following 10 combinations can be made

(3.3.3)	(3.3.5)	(3.5.5)	(5.5.5)
(3.3.7)	(3.5.7)	(5.5.7)	
(3.7.7)	(5.7.7)		
(7.7.7)			

The First Organ for Euler's Genera

Shortly before the war, the *Geluidstichting* was founded in the Netherlands, foundation which dedicated itself, in addition to the fight against noise, to ›ther acoustic problems. The *Geluidstichting* granted the necessary funds for he building of a small organ, on which the ten *Genera* of three numbers each :ited above could be played. The organ was constructed by VAN LEEUWEN Bro- hers, organ builders in Leiderdorp, and installed in the Teyler Museum at Haar- em. It had ten stops, corresponding to the *Genera.* Each stop had twelve labial ›ipes, corresponding to twelve equal-sized white keys of the keyboard. By means ›f knobs each stop could be provided with wind. Three of the *Genera* have only :our notes; on the organ they were represented in the range of three octaves. Six of the *Genera* had six notes; they were represented in the range of two oc- taves. The tenth *Genus* numbered eight notes; for it only an octave and a half were required.

The pipes were so purely tuned according to the vibration ratio of each *Ge- nus* that the intervals sounded without beats. From the control mechanism it sobn appeared that, in the Genus (3.5.5) inadvertently, in the place of the ma- jor third 4:5 = 8:10 (= 386 cents), an extra large, although consonant third 7:9 (= 435 cents) was sounding.

The Competition of 1944

The merchant and silk manufacturer, Pieter TEYLER VAN DER HULST (1702-1778) was deeply interested in theological and scientific questions. Every week scholars met at his house, and problems of religion, art and science were discus-

sed. When, in his will, he provided for the establishment of a cultural foundation, he also arranged that the work of this circle should be continued by the work of two groups, of six members each. "Teyler's *Godgeleerd Genootschap"* was the name of one group; "Teyler's *Tweede Genootschap"* the other. In the second group, the art of drawing, poetry, history, numismatics, physics, and biology were represented. Up to the present day, these societies annually offer a prize competition. The prizewinning works are published in the monographs of the societies. In this manner many outstanding studies have been created, and have appeared in the monographs of the societies.

In 1943 the representatives of the foundation and members of the second society decided to offer a competition for a prize in music. Twenty compositions should be submitted for the organ in the Teyler Museum, two for each of Euler's *Genera,* each piece to be not shorter than a half minute and not longer than ten minutes. As judges of the compositions, a jury of professionals was set up. The jury consisted of: Dr. A. SMIJERS, professor of musicology at the University of Utrecht; Dr. Jos. SMITS VAN WAESBERGHE, the learned medieval scholar; Eduard VAN BEINUM, the celebrated conductor of the Concertgebouw Orchestra in Amsterdam; Willem ANDRIESSEN, pianist and director of the Conservatory in Amsterdam; and Willem PIJPER, the outstanding contemporary composer of the Netherlands.

An Arithmetical Consideration of Music

The lectures and performances on the organ led me to more articles, and the publisher J. NOORDUYN EN ZOON N. V. in Gorinchem asked me to write a book for their scientific series on the physical and mathematical bases of music theory The editor of Huygen's papers, Dr. E. J. DIJKSTERHUIS, had already encouraged me to undertake a systematic treatment of the material. I agreed and took the book under consideration. Of especial value to me in this were the consultations with Martinus J. LÜRSEN, already named chief instructor at the Royal Conservatory in the Hague. When I showed the completed chapters to Willem Pijper, he advised me to provide the manuscript with an introduction to the subject matter. For this introduction I chose four historical sketches on the most important predecessors, Gioseffo Zarlino, Jean Philippe Rameau, Giuseppe Tartini and Leonhard Euler. In the final chapter of the book I gave my reasons for writing on music as best I, a physicist, could do.

In the summer of 1944 the book was completed. Because of the shortage of coal it could not be published. It appeared the next summer, 1945, under the title *Rekenkundige Bespiegeling der Muziek* published by J. Noorduyn en Zoon N. V., Gorinchem.

Modi antichi, Musiche nuove

There were five entries in the open prize competition. The one titled *Modi antichi, Musiche nuove* was declared the winner. It developed that the composer was Martinus J. LÜRSEN. In August, 1945, at a public event, the prize was presented to him in the Aula of the Teyler Museum. Three friends of the laureate, Jos DE CLERCK (violin), Jan TEGEL (viola), and Carel VAN LEEUWEN BOOMKAMP (cello) performed some of the prizewinning compositions. *Modi antichi, Musiche nuove* was published in 1947 by DE ERVEN BOHN, Haarlem; the text also has an English translation.

English friends

I got to know Ll. S. LLOYD, through Balthazaar Van der Pol, who often journied to England because of his radio-telegraph work, and who also once gave a lecture there before a musicological group. Initially with Lloyd I encountered that reticence with which one customarily protects himself against unknown inventors. After a personal conversation in London that was altered. Lloyd in turn introduced me to Alexandre MACCLURE, a Scottish doctor in Wellington (Salop), whose greatest pleasure was to sing the polyphonic madrigals of the Tudor period with acquaintances and friends. He was not willing to lose this pleasure through the use of the tempered, therefore wrongly tuned piano. He had built for himself a reed organ on which 19 notes to the octave could be performed by the use of knobs. At the beginning MacClure tuned the 19 notes after the manner of Woolhouse, in equal temperament, and conceived the smallest step as a third-tone of 63 cents, which, now with the doubled value of 126 cents should assume the role of the half-step, but turned out to be unsatisfactory. In addition, the fifth (with 694 instead of 702 cents) and the third (with 379 instead of 386 cents) were about one-third of a comma too small. MacClure was not content with this tuning, and followed a suggestion of Robert SMITH, who in 1749 had recommended a tempering in which the fifth and the major sixth were made *equally rough*. That corresponded practically to a mean-tone temperament, in which the fifth is taken a quarter comma too small, the major sixth a quarter comma too large.

Dr. MacClure and I became good friends. MacClure had prepared yet another organ on which the 19 notes were performable through adjustable shifting. In addition to the five sharp notes *(c♯, d♯, f♯, g♯, a♯)*, stood the five flat notes *(d♭, e♭, g♭, a♭, b♭)* and finally also *f♭* and *c♭*. The organ was installed in the church of the Benedictine Monastery, Ampleforth Abbey in York. Dom Lawrence BEVENOT O. S. B., played and cared for it. Today it is in the musicological department of Edinburgh University.

Public Interest

Only a few people such as Pijper and Lürsen, Smits van Waesberghe and Van Beinum showed an interest. From the musicological society of the Netherlands, the *Vereniging voor Nederlandse Muziekgeschiedenis*, I encountered rejection. When I proposed that we should sometime at one of their sessions make a comparison of tunings, and perform compositions of the 16th and 17th centuries on a harpsichord tuned in the modern fashion and then on one tuned in the manner of the period, with mean-tone temperament, this proposal was rejected by the Board of Directors on the grounds that it was indeed too specialized a question, that it could arouse no interest. In 1948 on the occasion of the festival of the International Society for New Music, Mr. Rutger SCHOUTE of Radio VARA helped me organize a performance in Hilversum. Hans PHILIPS played variations of SWEELINCK on two differently tuned harpsichords; the one harpsichord was tuned in modern temperament, the other in mean-tone temperament Only two of those present, Mr. MacClure and I, had at that time heard anything of this sort. People were astonished at the great difference in the tunings and by the charm of the old mean-tone temperament.

Some of the participants in the festival, among them Alois HABA, came to Haarlem in order to inspect the organ in Teyler's Museum. Haba was very interested . He explained to me on this occasion that his quarter-tones were not always 50 cents in size; there were larger or smaller quarter-tones according to musical requirements. Haba's remarks pointed in the direction of tuning in dièses, and seemed to me to confirm their usefulness.

Building a 31-Note Organ

It now became a question of raising the money to finance building a large organ. The first contribution was subscribed by the *Hollandsche Maatschappij der Wetenschappen* of Haarlem. Several private individuals joined in. Finally the Prince Bernhard Fund and the Ministry of Culture followed.

I sketched a 31-step special keyboard, which would permit both the play of full chords and rapid movement; it is described on page 42. The execution was entrusted to the organ builder B. PELS EN ZOON in Alkmaar. In the course of the building, the *Stichting Nederlandse Muziekbelangen* granted sums which made it possible for us to furnish a smaller keyboard with the conventional seven white and five black keys in addition to the large keyboard with 31-notes to the octave. By means of switches we could choose 12 pipes from the 31 pipes and connect them to the smaller keyboard. In this manner the 9 *Genera* of Euler which contained 12 notes, the *Genera*

$$(3^3.5^2), (3^2.5^3), (3^3.7^2), (3^2.7^3), (5^3.7^2), (5^2.7^3), (3^2.5.7),$$
$$(3.5^2.7) \text{ and } (3.5.7^2),$$

could be played on the conventional keyboard with the conventional technique.

Further Work

Meanwhile I extended a chapter of my book into a new book which was translated into English and appeared under the title *Just Intonation and the Combination of Harmonic Diatonic Melodic Groups* published by Martinus NIJHOFF, the Hague in 1949. I had been advised that the horizontal line (melody) was of greater value than the vertical line (chords). Therefore I stressed the existence of melodic groups, for example, numerically 6:7:8:9:10, with the 8 as the center of the melodic-harmonic relationship, as the representative of the fundamental. As is well known, the major scale is a linking together of three such groups which have their centers in the tonic, the dominant, and the subdominant. Now, however, there were not only harmonic-melodic groups, but there were also sub-harmonic groups, numerically 1/6:1/7:1/8:1/9:1/10, with the sub-harmonic center 1/8[1].

I always linked two groups together, either with two harmonic centers at a distance of a basic interval (fifth, third or seventh), or alternatively with two sub-harmonic centers at the same interval. Finally I constructed as well linkages of two groups with one harmonic and one sub-harmonic center. With such an inversion of the groups, they might in some cases have two notes in common. From here the way was open to the old church modes, which, as is well known, are classified on the basis of two central notes, the final and the dominant.

I wanted to provide the text with examples, small four-part *a capella* pieces in the possible tonal combinations. Willem Pijper sent me his student, Jan VAN DIJK, from Rotterdam, who composed 21 useful, charming and instructive examples for the book. Van Dijk also wrote ten pieces in the simple *Genera* of Euler for the first little organ. These pieces were recorded on magnetic tape and served me as tonal examples for my lecture at the Congress of the International Society for Musicology, which took place in the autumn of 1949 in Basel.

In Basel I met Jacques HANDSCHIN. When I told him how we, in Hilversum, had demonstrated the difference in the tuning between two variously tuned harpsichords, he was delighted.

1) Fokker here is advancing an idea which is familiar in physics, but generally unknown or ignored in music theory. Physics allows for the possibility of both overtones and "under" tones. Music has chiefly taken account of the overtone system only. Fokker derives the minor triad from the inverse mathematical relationship, and that relationship turns out to be as useful as the direct relationship. Throughout the text, there are references to the harmonic (generally overtone) system, and to the sub-harmonic (for want of a better word) system which corresponds to the inverse relationship.– Tr.

The 31-Note Organ

Early in the summer of 1950 the firm B. Pels en Zoon delivered the organ. It was installed in the Teyler Museum at Haarlem. The large keyboard, the one provided with the 31-note special keyboard, has two manuals. On the first manual are a Quintadena 8' and Prestant 4', on the second manual, Salicional 8' and Rohrflöte 4'; on the pedal, which is likewise made for the 31-note temperament, are a Sub-bass 16' and Gedeckt 8'. The organ has in all 648 pipes.

The junior chief, Mr. B. J. A. PELS, who had a year before received his engineer's diploma as a technical physicist from the Technical High School in Delft, set up the 31-note tempering by means of an electrical tone generator, whose oscillation created an interference with the air pipe oscillation in an oscilloscope by means of Lissajous' figures. The basic 31-note octave, from which the notes in other octaves were tuned, was produced through a progress in major thirds. In 12-note temperament, the third third (*b*♯) coincides with the octave of the beginning note (*c*); in the 31-note temperament it is a diësis lower than *c*. The chain of thirds is closed *not* after the third, but only after the 31st third.

The deviation from the pure third, 5/4, which we wanted, was so slight that as a result of the acoustic fusion through the air, the sounding pipes could not produce any beats. Therefore the beats required by the 31-note temperament were produced by means of a sounding pipe and a soundless electric generator. In the cyclic metamorphosis of Lissajous' figures, the beats were visible on the oscilloscope. It was a question of *one* beat in approximately 5 to 8 seconds. My suggestion to subject not only the third, fifth, and seventh but also the octave to a correction, so that they too would participate in the shading of the temperament, was energetically rejected by Mr. Pels, Jr.

The Dedication of the Organ

Mr. Paul Christian VAN WESTERING, organist in Bloemendaal, a suburb of Haarlem, had already shown a keen interest in the little organ. After the installation of the large organ, he practiced on the special 31-note keyboard and soon achieved an astonishing dexterity in play on the two manuals with their 11 rows of keys ranged above one another.

On Friday, September 8, 1950, the organ was played publicly for the first time. Mr. Van Westering played Sweelinck and Hurlebusch and did an improvisation. About fifty invited guests were present, among them representatives of the press.

The First Concert

Messrs. Van Westering and Van Dijk went to work. Mr. Van Westering collected his improvisations together as *Six Inventions*. Mr. Van Dijk composed a *Musica per organo trentunisono* in five parts:

Part I: Four Pieces for Organ Solo;
Part II: Four Pieces for Organ and Strings;
Part III: Four Pieces for Organ and Various Instruments (oboe, flute, violin, viola);
Part IV: Concerto for Organ and Orchestra;
Part V: Song for Organ and Voice ad libitum.

The usual thirds and fifths were assigned to the added instruments; the organ added to them had the intervals and the harmonies of the seventh, 7/4. Here there was truly a genuinely contrasting "concerto".

On Monday, the 10th of September, 1951, the concert took place in the Teyler Museum. Because of limitations of available space, there were about fifty listeners invited. Unfortunately Willem PIJPER did not live to hear this concert. Also, my English friend Dr. MACCLURE died in 1950. In his place Dom Lawrence BEVENOT O. S. B. from Ampleforth Abbey was present.

Performed were Sweelinck's "Toccata in A Minor", Van Westering's "Six Inventions" and Van Dijk's *"Musica per organo trentunisono"* parts II, III and IV. The performers were Paul Christiaan Van Westering and Roel Riphagen (organ), Jan Hesmerg and Han Holsberger (violin), Bernard Weijs (viola), Eduard Biele (cello), Jodocus De Laat (contrabass), Frans Vester and Marius Ruysink (flute), Cor Coppens and Koen Van Slogteren (oboe). Van Dijk conducted his *Musica* himself.

The concert was no success. Apparently the program was too overburdened; in place of the new compositions it should have contained more works by Sweelinck and other classic masters. There was also something to complain about in the organ playing.

The radio station N. C. R. V. picked up the concert and broadcast it. Parts of the recording were later performed at a meeting of musicians and physicists at the Teyler Museum to which the *Geluidstichting* had invited them, with better success. At a time when litigation took place between the radio station and the musicians' union, this recording was erased as an act of vandalism. I arrived too late to save it.

Further Performances

Monthly organ concerts followed the first public concert. Works from the 16th, 17th and 18th centuries were performed, played by organists from Haarlem and other cities. The twelve notes of the historic mean-tone temperament could be played on the conventional manual of the auxiliary keyboard so th it required no adjustment for the organist. Now and then Paul Christiaan Va Westering played on the large keyboard. Initially, he was the only one who mastered the special keyboard. Further there was still no literature for maki music in 31 notes.

Henk BADINGS now wrote a prelude and a fugue which indeed contained tain tonal refinements, but which could also be played on a conventional or After receiving his final diploma from the Amsterdam Conservatory, Anton BEER practiced and brought his technique so far as to play Badings' difficul pieces. To make a comparison, the radio company AVRO made a recording Badings' prelude and fugue on the 31-note Haarlem organ and on the 12-no organ of the Beginenhof chapel.

My friend Martinus LÜRSEN demonstrated the Haarlem organ to his coll gues and students of the Royal Conservatory of the Hague. Willem ANDRIES the famous pianist and director of the conservatory in Amsterdam, underto similar demonstration. When I asked him to play Sweelinck's *Chromatic Far sie* he was so struck by the difference between the large and small half steps the beginning measures of this work, indeed almost overwhelmed, that he p ed these measures over and again, and went no further.

It would be excessive here to enumerate all the demonstrations, performa and concerts. It should be mentioned further only that in 1952 the Internat nal Society for Musicology held its fifth congress in Utrecht and about 50 pa ticipants accepted my invitation to examine and hear the organ. All showed terest, quite a few were excited.

In 1955 I went into retirement and left Haarlem. The public organ conce ceased.

The Euler Celebration in Basel

Leonhard EULER, the Johann Sebastian Bach of mathematicians, was bor in Basel in 1707. In 1957 Basel celebrated the 250th birthday of its great so The director of the Euler collected edition, Professor Andreas SPEISER, kno not only as an outstanding mathematician but also as a sensitive art enthusia and friend of music, conceived the plan of honoring Euler's musical writing this memorial celebration, especially his *Tentamen novae theoriae musicae*, i special festival event. From Martin VOGEL he had learned that there was an gan in Haarlem on which one could perform the Euler *Musica Genera* with p sevenths, 7/4. He visited me for this purpose and discussed with me the prepara for an illustrated lecture for the Basel radio studio. The composers Henk BADI and Jan VAN DIJK would compose works in the Euler *Musica Genera*. The com sitions should be played on the Haarlem organ and be recorded by the Neth lands United Radio on tape, and should be sent to the Basel studio for use.

That took place. Besides the twenty works by Henk Badings and Jan Van Dijk, there were six pieces by Arie DE KLEIN[1], an amateur composer. The organist Bernard BARTELINK of Amsterdam played Badings' compositions; the organist Piet HALSEMA of Overveen played the remaining pieces. In Basel the arrangements were made through Mr. Konrad BECK. It took place on May 28, 1957. Dr. Martin Vogel and I gave the lectures. My lecture was punctuated with eight compositions; on the 3rd of June it was broadcast.

Paris and Marseilles

Professor Edmond BAUER, the outstanding physical chemist at the Sorbonne in Paris, showed great interest. As a young man he had almost devoted himself to writing music, but he renounced music for the scientific route. Bauer invited me to give a lecture in Paris. On the 15th of January, I spoke at the Sorbonne on *Les gammes et le tempérament égal* (The Scales and Equal Temperament) and showed pictures of the 31-note organ.

The American magazine *The Scientific Monthly* in October 1955 published an article on *Equal Temperament and the 31-keyed Organ.*

Professor Bauer again brought me into contact with Professor F. CARNAC who headed the Institute for Acoustical Research in Marseilles. Carnac invited me to speak before the International Colloquium of Music Acousticians which took place in Marseilles from May 27th through 29th, 1958.

In the same year, on October 11, 1958, I gave a lecture at the Netherlands Institute in Paris. Great interest was shown. Especially Dr. Max DEUTSCH, a pupil of Arnold Schönberg, appeared to be very convinced.

The Stichting Nauwluisterendheid

On February 15, 1960 a foundation was established in Haarlem which had as its goal to explore, to try out experimentally, and to propagate the further development of refinement of tonal steps. The foundation took the name *Nauwluisterendheid,* which one cannot easily translate. It has a double meaning. In the first place it means the act of totally precise, attentive listening; in the second place, it says that the question here is about things that go awry if one does not go about them knowledgeably, carefully and precisely.

The *Stichting Nauwluisterendheid* was subsidized by the government and organizations which supported music. Thanks to this help some good things quickly matured. From 1960 on, an organ concert takes place on the first Sunday afternoon of every month in the Teyler Museum.

1) A pseudonym for Professor Fokker. – Tr.

New works were composed, not only for the organ, but also for other instruments. As early as 1958 Hans KOX wrote three violin pieces in which the extra large whole step, 8/7, was dominant; Wim Stenz performed it. Further, Hans Kox composed a duo for voice (baritone) and violin; performers were Guus Hoekman and Wim Stenz. In 1961 Kox wrote four pieces for string quartet which were then performed on the radio Station N. C. R. V. on October 17, 1962. Performers were Wim Stenz, Walter Ellegiers, Jan Tegel, and Henk Lambooy. When Dr. Martin VOGEL spoke in Hilversum about the intonation problems of the contemporary orchestra and explained the brass instruments which he had developed, two enharmonically tunable trumpets were purchased, for which Hans Kox composed in 1964.

The English composer Alan RIDOUT wrote a trio for strings; Jan VAN DIJK wrote a trio for violin, cello and trombone. Both works have not yet been performed. A partita for cello by Ridout has been performed by Anner Bijlsma.

A vocal quintet was established which, among other things, brought to performance GESUALDO's chromatic madrigals and Henk BADINGS' *Contrasts.* In this composition, written in 1951, Badings utilized the harmonics and sub-harmonics from the eighth to the sixteenth overtones. The vocal quintet consists of Hea Ruys-Landheer, Nini Keularts-De Vries, Gerard De Vos, Frans Müller and Piet Van der Meulen.

Early in the year 1963, Peter SCHAT studied the use of tone clusters on the 31-note organ. His three organ pieces with the title *Collages* were performed in the Haarlem summer series for organ; they were played by Frans Van Doorn.

A sonata for two violins, which Henk BADINGS composed in the 31-note system, was performed in Amsterdam on February 1, 1964 with great success. It was played by Bouw Lemkes and his wife, Jeanne Vos.

Dr. Joel MANDELBAUM from New York made himself familiar with the orga in Haarlem in 1963 and wrote ten études for the simpler Euler *Genera Musica:* Anton de Beer played these pieces in an organ concert on March 1, 1964.

On October 14, 1965 four didactic pieces for two enharmonic trumpets and trombone by Hans KOX were performed in the Renaissance Room of the Fran Hals Museum in Haarlem. Performers were Harry Sevenstern and J. A. S. Doet (trumpet), Dick Leurink (trombone).

In 1961 Anton DE BEER wrote an organ method for playing on the 31-note-keyboard.

In 1964 Alsbach and Company published a book of exercises by me under the title *Oor en Stem*, solfège studies which contained the higher harmonics, especially those with the pure seventh, 7/4.

The *Mededelingen XXXI*, the publications of the *Stichting Nauwluisterendheid,* appear six times a year.

On March 17, 1966 the name of the foundation was changed. It is now called the *Stichting Huygens-Fokker.* In the future as well it will endeavor to re-

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arch all the possibilities for the construction of a new music which may differ om fashionable trends in the direction of atonal and antitonal experiments.

31-NOTE MUSIC SINCE 1966

by *Anton DE BEER*

In 1966, Bouw LEMKES wrote a pedagogic course of *Intonation exercises for two violins.* In that year also, Henk BADINGS completed a string quartet written in the 31-tone temperament, and Professor Fokker was appointed honorary member of the *Nederlands Acoustisch Genootschap* (the Netherlands Acoustics Society).

In 1967, to mark Professor Fokker's 80th birthday, Henk BADINGS wrote another Sonata for Two Violins. The work was first performed by Bouw Lemkes and Jeanne Vos. In that year Professor Fokker, producer/composer Pierre Abbink Spaink, Hans Kox, Joel Mandelbaum, Bouw Lemkes and his wife Jeanne Lemkes-Vos and Anton de Beer took part in a comprehensive interview on the development and the future of 31-tone music broadcast on Dutch radio. Works written in the 31-tone idiom were broadcast by RIAS Berlin, SDR Stuttgart, WDR Cologne and DR Copenhagen.

In the same year a first contact was established with Professor Leigh GERDINE, President of Webster College, St. Louis Missouri. This, together with the release in the United States by Professor Gerdine of a record featuring works by Badings, Kox and Ridout led to a heightening of interest in 31-tone music in the United States.

Arrangements were made with the Executive of the International Conductors' Course, an annual event organized by the N. O. S. (the Netherlands Broadcasting Foundation) in Hilversum, to incorporate two three-hour lectures on Just Intonation and the development of 31-tone music. Since then these lectures have become part and parcel of the course.

In 1968, the organ concerts were extended to include music of the past in a temperament which allowed the playing of pure thirds. The move appears to have caused widespread interest. The performing artists involved were: Bouw Lemkes and Jeanne Vos, violins, Nelly Duin, viola da gamba, and Anton de Beer, harpsichord.

Joel MANDELBAUM completed his opera *The Dybbuk* using the 31-tone technique in 1968. A scene from this opera has been performed several times by the vocal quintet Gesualdo di Venosa, which includes Hea Verhagen, sopraıo, Catherine Hessels, contralto, Gerard de Vos, male alto, Frans Müller, tenor, ınd Piet van der Meulen, bass, accompanied by Bouw Lemkes and Jeanne Vos, ʼiolins, and Anton de Beer, organ.

On February 2, 1969, the Utrecht Student String Quartet presented the first ›erformance of the new Badings String Quartet. On December 7th of the same ʼear, the work was played for the first time by the internationally famous Amai String Quartet which has since performed it at various concerts during their ours of Europe and America.

On November 20th followed the world premiere of the Concerto for Two /iolins and Orchestra by Henk BADINGS. It was performed by the Radio Philıarmonic Orchestra of the Netherlands Broadcasting Foundation in Hilversum vith Bouw Lemkes and Jeanne Vos, violins, under the baton of Charles de Volff. The work received further public performances in Brussels by the Bel;ian Radio and Television Orchestra conducted by Daniel Sternefeld and in Jtrecht, Holland and Bergen, Norway by the Utrecht Municipal Orchestra conlucted by Paul Hupperts.

A Swiss composer, Eugen FRISCHKNECHT wrote three works for 31-tone or;an which were performed several times. In this year, 1969, Professor Fokker .nd Hans Kox explained some aspects of the development of 31-tone music at he home of Holland's Crown Princess Beatrix and her husband, Prince Claus. 3ouw Lemkes and Jeanne Vos performed a number of compositions including ;erenade for Two Violins by Hans Kox.

In 1970, a new electronic 31-tone instrument was commissioned. It was :hristened the Archiphone, after the archicembalo, a cembalo built in Italy in 555, featuring 6 manuals with 31 tones per octave. Anton de Beer inaugurated he instrument at a concert in Teyler's Museum, Haarlem with a work he wrote specially for the occasion, *Intrada 1 November 1970.*

The archiphone was built by Hendrik VAN DER HORST, manufacturer of elecronic musical instruments in Wilp, in the province of Gelderland, Holland. The naker of the archiphone began designing the instrument back in 1966. The ıuilding of the instrument was made possible with the aid of grants from the)utch government and the Professor Lorentz Fund. The instrument was used, mong other things, in the performances of the scene from *The Dybbuk* by oel Mandelbaum. A series of demonstrations and concerts at Dutch conservatoies followed.

In 1971, a Dutch composer, Ton DE KRUYF, wrote a radiophonic composition *?UAHQUAHTINCHAN in den vreemde* (Q. . . . in foreign lands) which was ubmitted as the Dutch entry in the 1973 *Prix d'Italia* contest. The archiphone, ılayed by Anton de Beer, features prominently in the work.

Another work written in the 31-tone temperament is the *Speelmuziek* (Playul Music) by Anton DE BEER. The composition was first performed in the same ʼear by Bouw Lemkes, Jeanne Vos, violins, with the composer at the archiphone.

Anton de Beer wrote a number of works for the archiphone especially to demonstrate the potential of the instrument.

While staying in the Netherlands, Oedoen PARTOS, from Israel, wrote Three Fantasies for Two Violins. The work was commissioned by the *Oscar van Leer Foundation* and was first performed on May 18, 1972 by Bouw Lemkes and Jeanne Vos. On March 5, 1972, a Concerto for Violin, Cello and Trombone by Jan VAN DIJK received its first performance by Bouw Lemkes, Rene van Ast and Kees van Hage. Joel MANDELBAUM's settings of texts by Judy Berman, Three Songs for Soprano, Two Violins and Archiphone, were published the same year.

Professor Fokker died on September 24, 1972 at the age of 85. In him, the international world of music lost a striking personality who made a major contribution to the study of small intervals. His spirit will be kept alive in the Foundation that bears his name and which now has its own home at Haarlem, Holland.

SYSTEMATIC SECTION

Mean-tone Tuning

From the middle ages, after Guido d'AREZZO, people have sung the seven notes of the contemporary major scale with the syllables

ut *re* *mi* *fa* *sol* *la* *si*[1].

According to the then prevalent music theory, these notes stood in the tonal relationship of Pythagorean tuning, which can be obtained in the simplest manner through a progression in fifths or fourths which groups the seven syllables in the following series:

fa *ut* *sol* *re* *la* *mi* *si.*

Fifths and fourths counted as consonant, thirds and sixths as dissonant. Especially forbidden was the tritone *fa* to *si*. Within this fifth-fourth tuning, the major third corresponded to the numerical relationship 64:81, the tritone to the relationship 512:729. As, coming from the north, from England through the Netherlands, polyphonic song spread over Europe, through polyphonic usage was established the pure third 4:5 which is smaller than the dissonant Pythagorean third 64:81 by the comma 80:81. Toward the end of the 15th century, the century-long dispute over the consonance of the third was settled. With the recognition of the pure third 4:5, one was now confronted with the intonation problem of notes varying by a comma. Likewise, the century-long attempts and experiments with tempering the system began.

It is generally understood that three pure major chords and three pure minor chords can be built from the seven notes of the major scale, the major chords

1) Guido, of course, had only the first six syllables; the *si* (*ti* in current American usage) was added much later. Contemporary American practice also substitutes *do* for Guido's *ut*. – Tr.

la   mi   si

fa   ut,   ut   sol,   sol   re[1]

and the minor chords,

re   la   la   mi   mi   si

     fa,   ut,   sol.

But anyone who wants to tune an organ or a piano in this manner encounters difficulties. The *re* of the third major chord does not correspond to the *re* of the first minor chord; it is about a comma 80:81 higher. If one puts together the named chords in a tonal table, it is evident that we are concerned with a *re* which is really two different notes. The one *re* is the fifth above *sol*; the other is the fifth below *la*:

re   la   mi   si

     fa   ut   sol   re.

In mean-tone temperament a compromise is made in such manner that in place of the two versions of *re*, one is chosen which lies in between. How the other tones must fit with this agreement, let the tonal table above demonstrate. It has two horizontal lines and five vertical columns. The *re* of the first and that of the last column stand at the interval of a comma 80:81. If now this interval is removed, the four intervals from column to column are each diminished approximately the fourth part of a comma. The fifth thus becomes about a fourth of a comma diminished (1/4 in 80, that is 1 in 320), the fourths are about a fourth-comma augmented. The fifths 2:3 (= 320:480) become 321:480, the fourths 3:4 (= 240:320) become 240:321. This slight mistuning of fifths and fourths brought beats with it. They were accepted in the bargain, because one gained the pure third.

The pure third *ut-mi* was split into two like halves by the tempered *re*. The *re* stood in the middle between *ut* and *mi*. One spoke therefore of "mean-tone temperament." In this temperament, the fifth *ut-sol* is a fourth of a comma too

1) Note here that fifths are expressed in this table with horizontal distances, thirds by vertical distances. – Tr.

small, the sixth *ut-la* is a fourth of a comma larger than the pure sixth 3:5. Instructions were given therefore to make these intervals "equally rough." "Equa ly rough," however, is not precise enough. In order to tune an organ or a harp sichord in mean-tone temperament, one must take into consideration the resu tant beats.

If for *ut re mi fa sol la si* we substitute *c d e f g a b*, and we tune the *a* to 22 vibrations, then the pure fifth on *a*, the *e'* must have 330 vibrations. The third overtone of *a*, *e''*, at 660 vibrations, falls then together with the second overto of *e'*. However *e'* should be one vibration in 320 lower. The same is true for it: octave. Consequently *e''* should be 660/320 = 2.06 vibrations lower than 660. Against 660, one will then hear beats with a frequency of 2.06 per second, tha is 124 per minute. These beats can be checked with a metronome.

With the fifth *d-a* one orients himself by means of the common overtone *a'* at 440 vibrations. The beats here should have the frequency 440/320 = 1.375 vibrations, that is 82.5 per minute. One can check this similarly with the metr nome.

To tune the third below – *f* – correctly, one seeks the common overtone o *f* and *a*, namely *a''* with the frequency 880, and makes the beats disappear. In similar fashion one proceeds to tune the *c*♯ correctly from *a*.

From these basic rules a complete procedure for tuning is developed, by which mean-tone temperament can be done with the help of a metronome. This method was successfully used in Hilversum.

From 7 to 31

We now set about increasing the tonal range of the seven mean-tone tuned notes

f   c   g   d   a   e   b

and seek to broaden it through further fifths and major thirds. The third above and the fifth above on *f*, *c*, and *g* are already available. The first new note is *f*♯ to which *c*♯, *g*♯ and *d*♯ are added:

                f♯   c♯   g♯   d♯

f   c   g   d   a   e   b

Similarly with the thirds below and the fifths below. To the notes *a*, *e*, and *b* they are already available; *b*♭, *e*♭, *a*♭, and *d*♭ are still to be added, so that the tonal table now contains the following fifteen notes:

			f♯	c♯	g♯	d♯
f	c	g	d	a	e	b
d♭	a♭	e♭	b♭			

The conventional keyboard has only twelve notes. One must therefore decide to eliminate three notes. In the 17th century for the most part this meant that *d*♭, *a*♭, and *d*♯ had to be omitted.

Voices, however, are unlimited in their pitch possibilities. For voices one can therefore further complete the table, so that the middle column may have as many notes as the middle line.

			c♯♯	g♯♯	d♯♯	a♯♯
			a♯	e♯	b♯	f♯♯
			f♯	c♯	g♯	d♯
f	c	g	d	a	e	b
d♭	a♭	e♭	b♭			
b♭♭	f♭	c♭	g♭			
g♭♭	d♭♭	a♭♭	e♭♭			

Should one wish to go still farther, precise examination shows that with *e*♯♯, we have reached approximately the same pitch as the *g*♭♭ has; *c*♭♭ would coincide with *a*♯♯. To get from *g*♭♭ to *e*♯♯ one must cross over through three fifths to the right and seven major thirds up, and finally deduct four octaves. The difference would be

$$\frac{c\sharp\sharp}{g\flat\flat} = \frac{3^3 \times 5^7}{2^{21}} = \frac{2\,109\,375}{2\,097\,152}, \text{ approximately } \frac{173}{172}.$$

That would correspond to an interval of less than a half-comma. If one considers that in mean-tone temperament, in which we have found ourselves, the three

first steps of a fifth from *g*♭♭ via *d*♭♭ and *a*♭♭ to *e*♭♭ have all together three quarters of a comma too little, that is changed in this system so that the prio excess of 1/2 comma converts into a deficit of only 1/4 of a comma. One is j tified in describing the system contained in the table of fifths and thirds as cl sed in itself and complete.

That was and still is a great discovery. In the table one notes two squares e with sixteen notes, and one common corner. That thus creates 31 different n tes.

The system exists in a closed series. Any chosen interval, whether it be the fifth, major third, or minor third will lead back to the beginning note if repea ted 31 times. The system is satisfactory thus for all requirements for free trai position, while it contains the established beauty of the triad.

A characteristic of the system is that, within the minor thirds, it produces synonyms. Between *a* and *c* is *a*♯♯ = *c*♭♭, between *b* and *d* is *b*♯♯ = *d*♭♭. Further there are *d*♯♯ = *f*♭♭ and *e*♯♯ = *g*♭♭. Finally one can also designate *c*♯♯ = *e*♭♭♭, and others similarly.

The smallest interval contained in the system is the fifth of a step, of whicl there are 31 to the octave. These are dĭeses.

Small half steps (*c*-*c*♯ or *a*♭ to *a*) contain two dĭeses; large half steps or sma seconds (*e*-*f* or *g* to *a*♭) contain three dĭeses. The minor and the major thirds have eight and ten dĭeses, respectively; consequently, the fifths have 18 and the fourths 13 dĭeses.

The Pure Seventh

The mean-tone system came into being as people became accustomed to re place the Pythagorean third 64:81 with the pure major third 4:5 = 64:80. Th comma 80:81 was eliminated through the use of mean-tone temperament. In similar manner, the pure seventh, the natural seventh 4:7, can be included in the tonal system. The sum of a second 8:9 and two major thirds has the resul tant

$$\frac{8}{9} \times \frac{4}{5} \times \frac{4}{5} = \frac{128}{225}$$

The pure seventh has the resultant 4:7 = 128:224. The difference consists of o 224:225, very little more than 1 in 240 (1/3 in 80), that is, one third of a cor ma. Logically one can equate these two intervals with one another. This equating is more easily justified in the mean-tone system, as here the second is a h comma too small and therefore the sum of a second and two major thirds approaches the pure seventh more nearly; it is now not more than a third of a comma too large, but rather a sixth of a comma smaller than the seventh 4:7.

A glance at the table shows us thus that intervals such as *c* to *a*♯, *g* to *e*♯, *d*

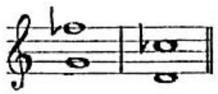
)

ı b♯, a♭ to f♯, e♭ to c♯, etc., can be viewed as pure sevenths.

One is easily convinced that this seventh, the sum of a second and two major iirds, 5 + (2 x 10) = 25 diëses. Its complement is the supra second with six .ëses.

The Notation of the Pure Seventh

Musicians schooled in conventional music theory will object that *g* to *e*♯ or to *b*♯ is no seventh, but rather an altered sixth, strongly dissonant in the bar-ıin, as the tonal relationship 128:225 shows. This objection we can negate if ·e furnish the pure seventh with its own form of notation. For that purpose ·e will use the little hook designed by TARTINI, which indicates a small amount f flatting, a "semi-flat."



ɩn *f* with a semi-flat sign we express in German as *fèh* with the sound *-èh* added ɔ the sound of the syllable in the same way as *"es"* is added in German[1]. The ıtervals *g* to *fèh* (*f* semi-flat) and *d* to *cèh* (*c* semi-flat) above can no longer be ıisunderstood as altered sixths. Sometimes one also needs three halves flat to ·e able to indicate the lowering of a half step plus a semi-flat flat. For this TAR-'INI also had a sign, made from the little hook and the usual flat note used to-ether.

Ve express this: *f* to *e* sesqui-flat, *c* to *b* sesqui-flat, etc. The whole step is thus ıow subdivided into five parts, which reckoned downwards for the whole step : to *f*, proceeds in the steps

g g½♭ g♭ g♭½ g♭♭ f[2]

.) There are vocabulary differences between English and German, and the text is altered to nake the case as clear as possible to English readers. – Tr.

!) The German version is *g, gèh, ges, gesèh, f*. Note that *g*½♭ is to be read as "*g* semi-flat". ;imilarly, g♭½ will be used to indicate "*g* sesqui-flat". The sharps will be treated similarly. – Tr.

Now we still need signs for the seventh below, which likewise runs the danger of confusion with the altered sixth. We take here the sign for the raising of a half step, and develop from the sharp the semi-sharp, the sharp with only one stem.

Instead of *f* flat – *d'*, *c'* flat – *a'*, *e* double flat – *c'*, we write and we say *e* semi-sharp – *d'*, *b* semi-sharp *a'*, *d* semi-sharp – *c"*[1]. Variously one makes use also of "three halves sharp" (a sesqui-sharp); e. g. instead of *d* flat to *b*, *g* flat to *e'*, *a* flat to *f*♯' we now write *c* sesqui-sharp – *b* (German *h*), *f* sesqui-sharp – *e'*, *g* sesqui-sharp – *f*♯'.

Thus the seventh below *f* sharp is *g* sesqui-sharp while *g* sesqui-flat is the seventh above *a* flat. Between *f* and *g* we have thus a rising series of five steps of the diësis:

f f½♯ f♯ f♯½ f♯♯ g[2].

Within the whole step *f–g* reckoned upwards

f g♭♭ g♭½ g♭ g½♭ g[3].

It is here clarified that synonyms are used for various signs of notation: *f*½♯ is identical with g♭♭, *f*♯ with g♭½, *f*♯½ with g♭, *f*♯♯ with g½♭. To this equivalency should be added *e*½♯ = *f*♭ and *e*♯ = *f*½♭. When one becomes accustomed to these equivalencies, one can get rid of the „sesqui-sharp" and „sesqui-flat". The whole step *f–g* thus divides itself into the tonal steps

1) The German version would be: *"fes–d', ces'–a', eses–c'. . . . . . ei–d', hi–a', di–c'."* – Tr.

2) The German version – *f, fi, fis, fisi, fisis, g*. Note here the distinction between *f*½♯ (*f* semi-sharp) and *f*♯½ (*f* sesqui-sharp). – Tr.

3) The German version – *f, geses, gesèh, ges, gèh, g*. – Tr.

f   f½♯   f♯   g♭   g½♭   g[1]

This method of notation is more comfortable. It is simpler to write *a*♭ to *f*♯ than g♯½ to *f*♯ or *a*♭ to g♭½. They are however the same tonal steps.

Tempered Mean-tone Tuning

Above we were concerned with trifling differences, with small fractions of a comma, with the purpose of getting rid of them. That now takes place through a regular temperament which one earlier would have called "equal." By definition the octave is divided into 31 equal steps.

Against the 12-note temperament, the 31-note temperament is distinguished through two essential advantages: it looks to the classical past in that, through its pure thirds, it enlivens anew the major and minor sounds in their old beauty and strength; it looks to the future, in that it makes consonance and euphony richer through its pure seventh, and it stretches the horizons of harmony.

When Paul HINDEMITH, in his well-known manual, sought to define and classify the musically useful notes, he stopped before the natural seventh. He was afraid that as a consequence of the inclusion of the seventh, he would be reduced to chaos through the multitude of notes created. Hindemith obviously did not know the possibilities of the forgotten recommendation for 31-note temperament which would put everything into beautiful order and would dispel the feared chaos with only 31 notes.

It now becomes only a question of realizing the technical and musical possibilities and taking advantage of them.

The Manual

As is well known, on the ordinary keyboard the 12 notes per octave are organized into two rows. The lower row has seven white keys of similar width per octave. The other row, unequally spaced, has groups of two and three black keys which are raised and shoved back on the keyboard. It is clear that one cannot organize 31 keys into two rows without exceeding the reach of the hand. One must therefore decide to make use of more rows, indeed at least five, in order that one need not span more than six or seven keys with one hand.

In the creation of a new keyboard one should also take advantage of the opportunity to eliminate the unsatisfactory characteristics of the conventional keyboard. One should avoid that a given distance should at one time be a whole step and at another time a half step. We should avoid that the same interval, let us say a major third, should at one time be played with two notes from the

1) The German version – *f, fi, fis, ges, gèh, g*. – Tr.

same row, another time with notes from different rows.

We will therefore begin by ordering the keys for the whole steps in a horizontal row. The row consists of the notes *c, d, e, f*♯, *g*♯, *a*♯, *b*♯, *c*♯♯, *d*♯♯, *g*♭♭ *a*♭♭ *b*♭♭, *c*♭, *d*♭, *e*♭, *f*. In a second row, behind the first and higher, we will put between *e* and *f*♯ the *f*, and construct on each of the two sides of the *f* the whole-tone scale, *d*♭, *e*♭, *f*, *g*, *a*, *b*, *c*♯, *d*♯, *e*♯, *f*♯♯, *g*♯♯, *a*♯♯, *d*♭♭, *e*♭♭, *f*♭, *g*♭, *a*♭. In a third row, still further behind and higher, we will place *c* between *b* and *c*♯, and construct on both sides of the *c* the row *d*♭♭, *e*♭♭, *f*♭, *g*♭, *a*♭, *b*♭, *c*, *d*, *e*, *f*♯, *g*♯, *a*♯, *b*♯, *c*♯♯, *d*♯♯, *g*♭♭, *a*♭♭. Thus one continues to the twelfth row. The following schematic representation makes use of the briefest synonym for each note[1].

1) It should be clearly understood that the notes indicated here are not enharmonically interchangeable with notes in the 12-tone system; i. e. *b*♯ is not *c*, etc. The notes here mentioned are those derived from the schematic table on page 38. Their names, in the table which follows here, are altered according to Fokker's suggestion in the section titled "The Notation of the Pure Seventh" (page 40) where each of the dièses is given the simplest possible designation. – Tr.

With the 31-step keyboard, note the following:

1) The whole-tone intervals are always in a horizontal row.
2) The rising dieses always lie in vertically rising steps.
3) The rising large half-steps lie in a diagonal in steps rising to the right.
4) The rising small half-steps lie diagonally in steps descending to the right.
5) Three major thirds lead to a diesis less than the octave.
6) The notes whose names in German are without suffix have white keys (*c*, *f*).
7) The notes with the suffix "s" in German have black keys (*c*♯, *d*♭).
8) The notes with the suffix "i" or "h" in German have blue keys (*d*½♯, *e*½♯, *a*½♭).
9) For every note there are two keys available; for the notes of the first and the last row, even three. That simplifies the playing. Fingers and thumbs choose keys which lie most conveniently.
10) Every stretch of a given distance and length leads regularly to the same interval. In transposition one needs only move the hand, without altering the fingering.
11) The manual has space enough for the electrical contacts of the organ. For the pianoforte the problems of the changeover are still not resolved.
12) The pedal organ is organized similarly, but it is furnished with only five rows. For every note only one key is available. The tonal range of the pedal consists of an octave and a half, from *C* to *f*.
13) The key *d* has a central symmetrical position, considering the colors of the keys.

d e f♯ g♯ a♯ c½♭ d½♭ e½♭ f½♯ g½♯ a½♯ b½♯

d♯ f½♭ g½♭ a½♭ b½♭ c½♯ d½♯ e½♯ g♭ a♭ b♭

d½♭ e½♭ f½♯ g½♯ a½♯ b½♯ d♭ e♭ f g a b

d½♯ e½♯ g♭ a♭ b♭ c d e f♯ g♯ a♯

d♭ e♭ f g a b c♯ d♯ f½♭ g½♭ a½♭ b½♭

d e f♯ g♯ a♯ c½♭ d½♭ e½♭ f½♯ g½♯ a½♯

c♯ d♯ f½♭ g½♭ a½♭ b½♭ c½♯ d½♯ e½♯ g♭ a♭ b♭

d½♭ e½♭ f½♯ g½♯ a½♯ b½♯ d♭ e♭ f g a

c½♯ d½♯ e½♯ g♭ a♭ b♭ c d e f♯ g♯ a♯

d♭ e♭ f g a b c♯ d♯ f½♭ g½♭ a½♭

c d e f♯ g♯ a♯ c½♭ d½♭ e½♭ f½♯ g½♯ a½♯

The eleven rows continue to left and right, as far as the range of the organ ıches. The representation above is only a section, which does not extend two taves.

The German original of this table is given below. – Tr.

d e fis gis ais cèh dèh èh fi gi ai hi

dis fèh gèh àh hèh ci di ei ges as b

dèh èh fi gi ai hi des es f g a h

di ei ges as b c d e fis gis ais

des es f g a h cis dis fèh gèh àh hèh

d e fis gis ais cèh dèh èh fi gi ai

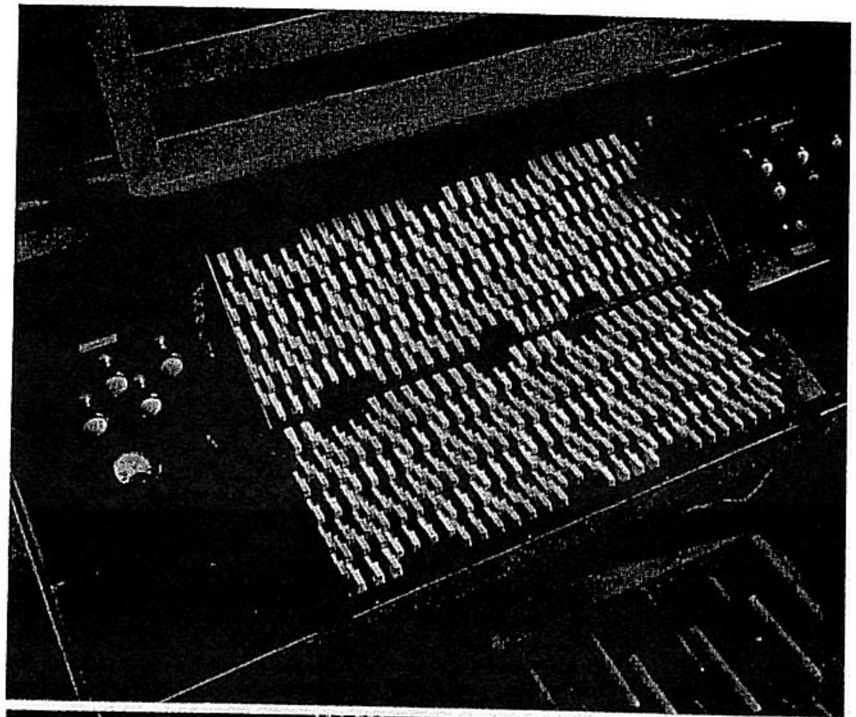
cis dis fèh gèh àh hèh ci di ei ges as b

dèh èh fi gi ai hi des es f g a

ci di ei ges as b c d e fis gis ais

des es f g a h cis dis fèh gèh àh

c d e fis gis ais cèh dèh èh fi gi ai



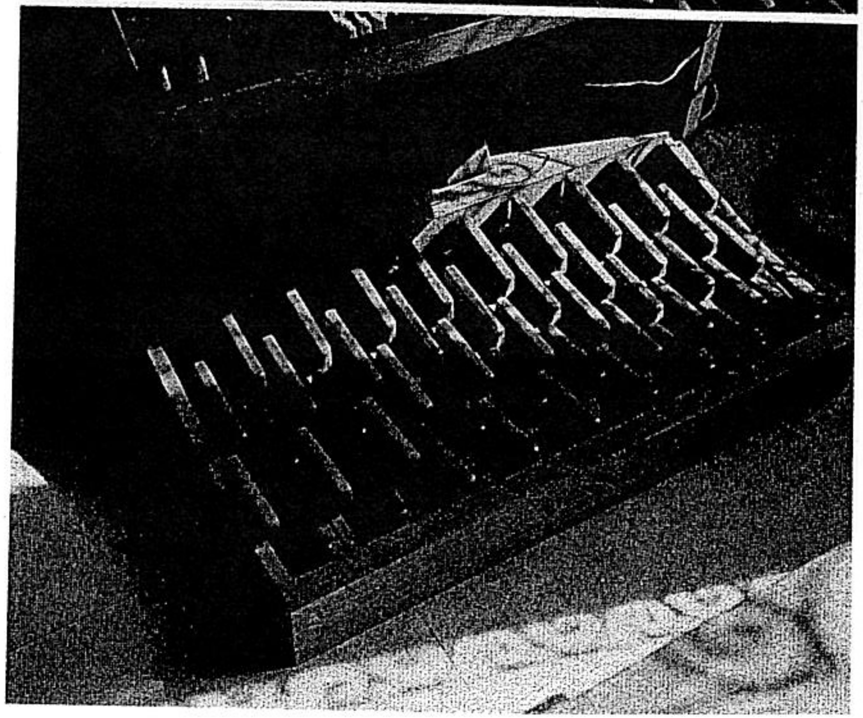


Table of intervals in the 31-note system[1]

	Ascending	*Descending*
C to C	Perfect Octave	Perfect Prime
C to C semi-flat	Infra octave	Supra prime
C to B semi-sharp	Supra seventh	Infra second
C to B	Major seventh	Minor second
C to B semi-flat	Mean seventh	Mean second
C to B flat	Minor seventh	Major second
C to A sharp	Augmented sixth	Diminished third
C to A semi-sharp	Supra sixth	Infra third
C to A	Major sixth	Minor third
C to A semi-flat	Mean sixth	Mean third
C to A flat	Minor sixth	Major third
C to G sharp	Augmented fifth	Diminished fourth
C to G semi-sharp	Supra fifth	Infra fourth
C to G	Perfect fifth	Perfect fourth
C to G semi-flat	Infra fifth	Supra fourth
C to G flat	Diminished fifth	Augmented fourth
C to F sharp	Augmented fourth	Diminished fifth
C to F semi-sharp	Supra fourth	Infra fifth
C to F	Perfect fourth	Perfect fifth
C to F semi-flat	Infra fourth	Supra fifth
C to E semi-sharp	Supra third	Infra sixth
C to E	Major third	Minor sixth
C to E semi-flat	Mean third	Mean sixth
C to E flat	Minor third	Major sixth
C to D sharp	Augmented second	Diminished seventh
C to D semi-sharp	Supra second	Infra seventh
C to D	Major second	Minor seventh
C to D semi-flat	Mean second	Mean seventh
C to D flat	Minor second	Major seventh
C to C sharp	Augmented prime	Diminished octave
C to C semi-sharp	Supra prime	Infra octave
C to C	Perfect prime	Perfect octave

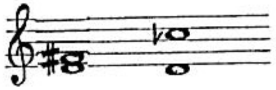
The Intervals

To begin with, the organ has all the required individual notes. Any pair of these create an interval. If both notes belong to the same overtone series – to the overtones of the same fundamental – we have an harmonic interval. Two numbers are associated with such an interval, namely the numbers of the coinciding upper harmonics. Intervals such as 2:3, 3:7, 9:11 are harmonic intervals. The two parts of an harmonic interval are always also sub-harmonics of a common overtone. The notes of a fifth (2:3) for example *d* and *a*, have besides their common fundamental *D* a common overtone in *a'*. Also related to them are the third and second sub-harmonics; their numerical relationship can be expressed through the inverse numbers 1/3:1/2. The same may be said of 3:7; 3 is 1/7 and 7 is 1/3 of the note 21, their common overtone.

EULER called such a common overtone the "exponent." I have, on certain specific grounds accustomed myself to call it the "guidetone." Every harmonic interval thus has a fundamental and a guidetone and indeed at the same distance

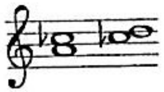
The simplest intervals with which we are chiefly concerned are obviously

1) This table of intervals in the 31-note system, suggested by Professor C. Kendall STALLINGS, is very helpful and essentially simple. One of its great virtues is that it retains all of the major and minor, augmented and diminished intervals as they are, with their invertible characteristics unchanged. It adds the mean intervals, which invert into other mean intervals, and the supra and infra intervals, which invert reciprocally. – Tr.



4:5 4:7

5:6 5:7 5:8



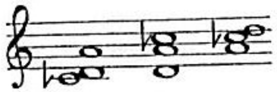
6:7 7:8

Three-Note Chords[1]

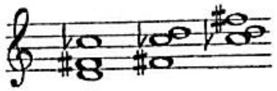
Through the linkage of two harmonic intervals one can create a great number ıf harmonic chords. From this store we choose the simplest which, omitting the 'actor 2, consist of two or three which contain the factors 3, 5, and 7. Those are:

3:4:5 4:5:6 5:6:8

7:8:12 4:6:7 6:7:8



4:5:7 5:7:8 7:8:10



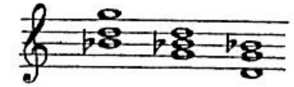
5:6:7 6:7:10 7:10:12

) Fokker here uses the German word, *Dreiklänge*, which means to him a chord (or com-lex) of three members. Throughout the passage which follows, the word "triad" or "chord" . to be re-interpreted freely in this sense, understanding that in Fokker's new system the :rm "triad" is scarcely applicable; "trichord" is more nearly acceptable; "complex of three ıembers" would be more accurate, but too cumbersome. – Tr.

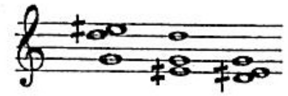
Measured according to range, in the most compressed form, there are chords *within the fifth* (4:5:6), *within the fourth* (6:7:8), *within the diminished fifth* (7:8:10), and *within the augmented fourth* (5:6:7). The latter ones contain no representative of their fundamental.

All chords can be mirrored, or stated otherwise, inverted, turned upside down. Then the following chords are created:

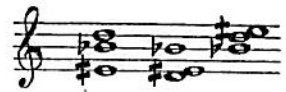
$\frac{1}{3}:\frac{1}{4}:\frac{1}{5}$ $\frac{1}{4}:\frac{1}{5}:\frac{1}{6}$ $\frac{1}{5}:\frac{1}{6}:\frac{1}{8}$



$\frac{1}{7}:\frac{1}{8}:\frac{1}{12}$ $\frac{1}{4}:\frac{1}{6}:\frac{1}{7}$ $\frac{1}{6}:\frac{1}{7}:\frac{1}{8}$



$\frac{1}{4}:\frac{1}{5}:\frac{1}{7}$ $\frac{1}{5}:\frac{1}{7}:\frac{1}{8}$ $\frac{1}{7}:\frac{1}{8}:\frac{1}{10}$



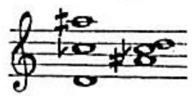
$\frac{1}{5}:\frac{1}{6}:\frac{1}{7}$ $\frac{1}{6}:\frac{1}{7}:\frac{1}{10}$ $\frac{1}{7}:\frac{1}{10}:\frac{1}{12}$

To every harmonic chord there is an opposing sub-harmonic chord. They mirror one another. One can also create chords through the repetitions of the same interval. Such chords are identical with their inversions:

4:6:9 9:12:16

16:20:25 25:40:64

16:28:49 49:56:64



The Linkage of Two Three-Note Chords

One can join together two chords of the same kind which have two common tones into a chord of four notes. For that purpose, these chords must be inverted reciprocally (or mirrored).

The chords *within the fifth* can be mirrored in three ways with two common tones. Schematically we can show that through the figures represented in this diagram:

f# | | f#——c# | | f#——c#
d——a | with | a | gives | d——a
8:10:12 | | 10:12:15 | |

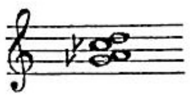
f# | | b——f# | | b——f#
d——a | with | d | gives | d——a
12:15:18 | | 10:12:15 | |

f# | | d——a | | f#
d——a | with | f | gives | d——a
20:25:30 | | 20:24:30 | | f

In the first case one gets a closed rectangle. We call it a closed four-note chord and designate it with (3, 5). To the triad *d–f#–a* is added *c#*, the leading tone to the fundamental *D*. In the second and third cases above, the structures are open. In the second case it is the triad "with an added sixth" of RAMEAU; in the third case it is a four-note chord with a major and a minor third.

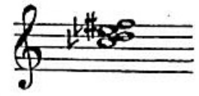
The chords *within the fourth* can similarly be mirrored and added to. For the moment let us confine our interest here only to the linkage of a closed four-note chord, which we designate (3, 7).

c½b——g½b | 28 | 21
d——a | 32 | 24



The chords *within the diminished fifth* similarly create a closed four-note chord; it is designated with (5, 7).

c½b——e½b | 28 | 35
d——f | 32 | 40



The Linkage of Three Three-Note Chords

Obviously the conventional major scale is produced from the linkage of three major triads, called the tonic, dominant, and sub-dominant. In *C* major the chords

e | b | a
c g | g d | f c
Tonic | Dominant and | Sub-dominant

create the *c* major scale

a e b
f c g d¹.

In the same manner the three minor triads may be put together into a scale

a e | e b | d a
c | g | f
Tonic | Minor dominant | Minor sub-dominant

1) Note that in these diagrams, Fokker represents intervals of a fifth in the horizontal rows and thirds in the vertical columns. In the mathematical statement, the number 3 represents the interval of a fifth, and the number 5 represents the interval of a third. -- Tr.

From the notes

d	a	e	b
	f	c	g

an *a* minor scale can be built: *a b c d e f g*. One should however note that the *d* here lies a comma below the *d* of the *c* major scale. We have already spoken about the difference between these two notes, *d*, and their compensation in mean-tone temperament.

It is worth noticing that the schematic figure is not closed. In the *c* major tonality the leading tones *b* and *e* are related to the tonic and the subdominant; the dominant lacks a corresponding leading tone. It is therefore necessary to supply this leading tone *f*♯ to the dominant. Thus one arrives at a closed harmonic framework,

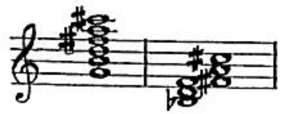
a	e	b	f♯
f	c	g	d

which consists of the three closed four-note chords joined together. It is therefore evident here that the *e* minor scale combines with the *c* major scale. In the finale of BEETHOVEN's E Minor String Quartet, one finds this combination of *e* minor and *C* major clearly. Such a structure we name, according to EULER, *Genus Musicum*. Symbolically we designate it (3^3. 5), because the closed four-note chord (3.5) was twice transposed upwards through the fifth (No. 3).

The Linkage of Two Four-Note Chords

If we transpose a closed chord of four notes through an interval which is contained within it, we can join the two four-note chords together in a similarly closed structure. Two closed four-note chords (3. 5) join together either as

b	f♯	c♯	or as	f♯	c♯
g	d	a		d	a
				b♭	f



Transpositions performed with 3 or 5 result in the formulary statement (3^2 or, respectively, (3.5^2). These structures should be called closed chords of s notes.

Two closed chords of four notes (3.7) can correspondingly be connected

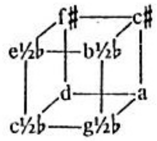
f½♭	c½♭	g½♭	or as	c½♭	g½♭
g	d	a		d	a
				c½♯	b½♯

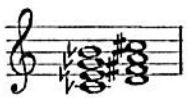
These are the closed six-note chords (3^2.7) and (3.7^2). The latter contains tl diësis *b*½♯ to *c*½♭.

Two closed four-note chords (5.7) lead to the construction of closed six-note chords (5^2.7) and (5.7^2):

a♭½	c½♭	e½♭	and	c½♭	e½♭
b♭	d	f♯		d	f♯
				f♭	a♭

A closed four-note chord can also be transposed through an interval which ıot contained within itself. If one transposes the chord (3.5) through a se- ıth, or (3.7) through a major third, the same result happens. There is created losed chord of eight notes with the formula (3.5.7). In the visual represen- ion, one must leave the two dimensional page. The closed chord of eight no- is represented by a cube, which obviously has six sides. The closed chord of ht notes contains three pairs of closed chords of four notes.





Carrying this thought through, there are harmonic structures which one l cannot call closed chords of six notes, because they have only four notes, t which also are not closed chords of four notes because they contain only ɛ interval (through repetition). One can call them geometric chords of four tes. As examples should be listed:

g	d	a	e,	Formula (3^3)
b♭	d	f♯	a♯,	Formula (5^3)
e½♯	d	c½b	a½♯.	Formula (7^3)

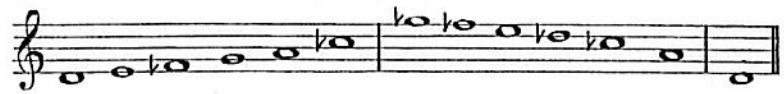
mmarizing, one can designate all structures whose formulae contain three tors within parentheses as Leonhard EULER's *Genera Musica* of the third ;ree. These are the simplest; there are more extended ones.

The Linkage of Three Four-Note Chords

As in the conventional music theory we connected three closed chords of four notes (3.5), similarly we can add to one another three closed chords of four notes (3.7)[1].

f½b c½b g½b d½b

g d a e

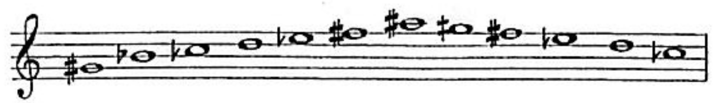


If a Gregorian chant in the second church mode is sung without the accompaniment of an organ, one hears the notes *re mi fa sol la do* (or *ut*) interpreted very often in the sense of the rising scale above. One notes the steep, downward directed leading tone *f½b* to *e* and the supra second between *f½b* and *g*. This structure, arising through a double transposition of the closed four-note chord (3.7) through a fifth (3), we call the *Genus Musicum* (3^3.7).

We now add to one another the three chords of four notes (5.7) through the transposition of a third.

g♯ c½b e½b g½♯

b♭ d f♯ a♯



The rising and the falling scales are mirror images. Their last note is not like the first note. This *Genus Musicum* is designated (5^3.7). Since we cannot here transpose by a fifth, we have utilized the major third for transposition.

1) Note that in this diagram, Fokker represents the sevenths vertically (the number 7 of the formula), and the fifths horizontally (the number 3). – Tr.

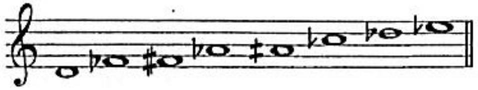
By transposition through the seventh the *Genus Musicum* (5.7^3) arises.

a½♯ d♭

c½♭ e½♭

d f♯

f♭ a♭



In this tonal series, *a*♭ to *d*♭ and *f*♭ to *a*½♯ can scarcely be distinguished from fourths. One should notice that *f*♭ is the same as *e*½♯. The difference amounts to about 5 in 1,000, which would be 2/5 of a comma. We find here a division of the fifth *a*½♯ to *f*♭ and *d*♭ to *a*♭ into three parts.

The closed chords of four notes (3.7) we can combine with one another through transposition at the seventh. Also here we find a three-part division of the fifth. *f*♭ equals *e*½♯.

a½♯ e½♯

c½♭ g½♭

d a

This *Genus Musicum* (3.7^3) permits the arrangement of a series of supra seconds, which is similar to the notes of the Javanese and Balinese *gamelan slendro.*

Finally we note that the closed four-note chords (3.5) can also be combined through a transposition of a major third

a♯ e♯

f♯ c♯

d a

b♭ f

The three major thirds in the *Genus musicum* (3.5^3) bring two steps of a diĕsis into the scale: *a*♯ to *b*♭ and *e*♯ to *f*.

With this we have assembled the harmonic structures which can be created out of the new chords using the model of traditional scales.

The Circled Mirrorings of a Three-Note Chord: Doubled Connections

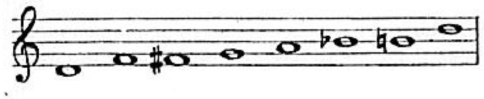
Useful harmonic structures are created by a repeated inversion of a triad, in which one note remains fixed and the other notes take over the role of a second connecting note or are exchanged against the notes created through mirroring.

One might begin with the triad within the fifth *d*–*f*♯–*a* = 4:5:6. *d* remains fixed. *f*♯ becomes a connecting note in the first mirroring, in which the *a*, the fifth above *d*, is exchanged for *b*, the fifth below *f*♯; this produces *b*–*d*–*f*♯. In the second inversion in addition to the *d*, the *b* should be taken as a connecting note. The *f*♯, the fifth above *b*, is mirrored in the fifth below *d*. This results in the triad *g*–*b*–*d*. With *g* and *d* as connecting notes through the renewed mirroring one creates *g*–*b*♭–*d*. Inversion with fixed *b*♭ and *d* leads to *b*♭–*d*–*f*. There follows, with fixed *d* and *f*, the triad *d*–*f*–*a*. The last mirroring finally leads through a fixed *d* and *a* back to the triad *d*–*f*♯–*a*. In six steps the circle is closed. Our method in this circular mirroring is clarified by the diagram of the fifths and thirds:

b f♯

g d a

b♭ f



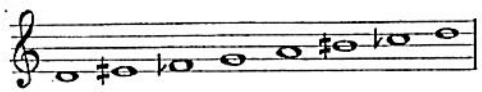
The seven notes create a chromatic scale of seven members which begins and ends with a minor third which is centrally symmetric around *d*; and which contains only intervals which are also used in conventional music theory. In classical music it has not yet been given attention.

We now carry out the circular mirroring of the chords within the fourth, using the chord *a–c½b–d* = 6:7:8. *d* again remains fixed in all the mirrorings. The first connecting note will be the *c½b*; thus the *a* which is a fourth below *d* will be mirrored above the fourth above *c½b*. This is *f½b*. In the second inversion of the new chord with three notes, *c½b–d–f½b*, in addition to the *d*, the new note *f½b* remains fixed. The *c½b*, the fourth below *f½b*, becomes a *g*, the fourth above *d*. In the third inversion *g* and *d* remain fixed. From *f½b* comes *e½♯*. The mirroring of the three-note chord *d–e½♯–g* creates, with a fixed *d* and *e½♯*, the three-note chord *b½♯–d–e½♯*. With a fixed *d* and *b½♯* this becomes *a–b½♯–d*. Finally the mirroring of *a* and *d* brings us back to the beginning chord *a–c½b–d*. In the area of the fifth and the seventh, the schema creates a circular mirroring as before

f½b c½b

g d a

e½♯ b½♯



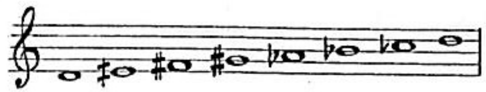
Again a seven-note scale is created. It arises from the tetrachords *d–g* and *a–d*. Both tetrachords have a diësis in their middles. We therefore speak of a seven-note bi-diësis scale.

In similar fashion the circular mirroring of the three-note chord within the diminished fifth, *b♯–d–f♯* = 7:8:10, is carried out. With a fixed *b♯* and *d* the *f♯* of the third above *d* is replaced by the *g♯* of the third below *b♯*. With *g♯* and *d* as fixed notes, the *b♯* is mirrored in *b♭*. The *g♯–b♭–d* is mirrored with a fixed *b♭* and *d*, in *f♭*. The inversion with the fixed *d* and *f♭* makes from *b♭–d–f♭* the three-note chord *d–f♭–a♭*. That again becomes *f♯* with fixed *d* and *a♭*, *d–f♯–a♭*. The final mirroring with a fixed *d* and *f♯* leads back to the beginning chord *b♯–d–f♯* and closes the circle. The diagram of this circular mirroring in the plane of the third and the seventh shows us these seven notes in their harmonic relationships:

g♯ b♯

b♭ d f♯

f♭ a♭



One should note that *f♭* = *e½♯* and *b♯* = *c½b*. *g♯* and *a♭* are at a distance of one diësis. We have thus a seven-note mono-diësis scale before us. If one were to remove the step of the diësis between *g♯* and *a♭* and let *f♭* equal *e* and *b♯* equal *c*, then this scale would degenerate to a six-note whole-tone scale.

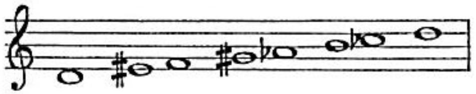
A special surprise is in store for us from the three-note chord within the augmented fourth, *b–d–e♯* = 5:6:7. We already said concerning it that it does not contain its own fundamental. If now in the mirroring the *d* is held constant, it never becomes the fundamental. With a fixed *d* and *e♯*, the *b*, the minor third below *d*, is mirrored in the minor third above *e♯*, *g♯*. Mirroring with a fixed *d* and *g♯* makes from *d–e♯–g♯* the three-note chord *d–f–g♯*. With a fixed *d* and *f*, the next inversion gives rise to the sound *c♭–d–f*, which again, with a fixed *c♭* and *d*, is altered to *a♭–c♭–d*. If the *a♭* and *d* are now fixed, the *c♭* is mirrored as *b*. The sixth inversion leads finally again back to the original chord: from *a♭–b–d* comes *b–d–e♯*. In the tonal relationship of this chord, the prime numbers 3, 5, 7 are contained.

We order the notes therefore in a series of minor thirds 6:5, tritones 7:5 and augmented seconds[1].

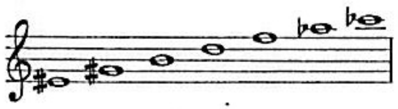
b ab

e♯ d c♭

g♯ f



This creates a seven-note three-diēsis scale. A chain of intervals built on six consecutive minor thirds can be constructed from it.



However it is to be observed that the thirds *g♯* to *b* and *f* to *a♭* stand in the relationship 21:25, therefore not in the relationship of the pure minor third 5:6. The tonal relationship of the diēsis *g♯–a♭* is 49:50, that of the two other diēses is 35:36.

Circular-Mirroring of a Three-Note Chord: Simple Couplings

In the mirroring of a three-note chord we need not necessarily utilize two constant notes; a single note can also be the center of inversion of the three-note chord. The notes of the three-note chord are indicated by three prime numbers. The center of the inversion is fixed by these numbers in succession. Every inversion will create two new notes. The closed circular mirroring with its six inversions will create a 12-note scale.

We begin with the three-note chord within the fifth, with *b–d♯–f♯*. The first inversion is built from *b* (prime number 2); this results in *e–g–b* (= 1/6: 1/5:1/4). The second inversion would follow from *g* (prime number 5); it leads to *e♭–g–b♭* (= 4:5:6). Next the inversion from *b* flat (prime number 3); one

1) Note that in the diagram minor thirds are expressed vertically, augmented seconds are expressed horizontally, and the tritone is expressed on a diagonal from left to right. – Tr.

creates *b♭–d♭–f* (= 1/6:1/5:1/4). Now again the prime number 2, thus the *f* in the series, is taken as the center of the inversion. Its mirroring leads to *f–a–c* (= 4:5:6). The fifth mirroring through *a* creates *f♯–a–c♯*, the sixth through *f♯* leads finally again to the beginning chord *b–d♯–f♯*. The ordering of these chords in a schema of fifths and thirds shows the following tonal arrangement

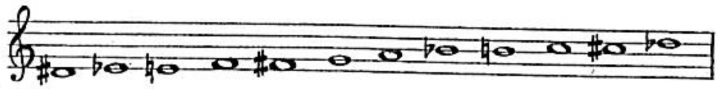
d♯

e b f♯ c♯

g a

e♭ b♭ f c

d♭



In the middle of the tonal scheme the *d* is missing. This unwritten note is the center of the harmonic symmetry of the 12 note bi-diēsis scale, which is chromatic throughout. The diēses *d♯–e♭* and *c♯–d♭* arise from the sequential use of three major thirds. There are further two series of three fifths and two series of three minor thirds.

We now undertake the same sort of circular mirroring of the three-note chord within the fourth, 6:7:8, *c½♭–d½♯–f½♭*. The center of inversion will first be the *f½♭*, which is represented by the prime number 2; the mirroring creates the chord *f½♭–g–b♭½* (= 1/8:1/7:1/6). Inversion on *g* (prime number 7) gives *e½♯–g–a½♯* (= 6:7:8). Inversion on *e½♯* (prime number 3) gives *b½♯–d½♭–e½♯* (= 1/8:1/7:1/6). Inversion on *b½♯* (prime number 2) gives *f♯½–a–b½♯* (= 6:7:8). Inversion on *a* (prime number 7) gives *g½♭–a–c½♭* (= 1/8:1/7:1/6). The sixth inversion through *c½♭* (prime number 3) leads back to the original chord *c½♭–d½♯–f½♭*. The disposition of the notes in the diagram of fifths and sevenths makes it clear precisely why this division of the circle was demonstrated on the chord *c½♭–d½♯–f½♭*. The notes group themselves in the tonal region of *d*, which is not present, but which nevertheless is to be regarded as the center of symmetry of this part of the circle.

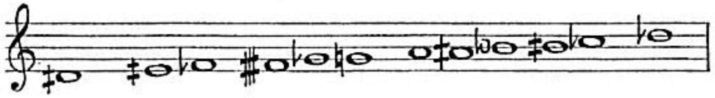
d½#

bb½ f½b c½b g½b

g a

a½# e½# b½# f#½

d½b



This twelve-note scale contains six dieses, of which two dieses follow each other directly twice. The dieses *g½b -g* and *a–a½#* stand in the relationship 63:64, the rest in the relationship 48:49. As the tonal diagram above clearly shows, there follow one another three fifths twice, the three supra seconds twice, and in the diagonal, three infra-thirds[1] (6:7). One should also note the two pseudo-fifths, *d½#* to *a½#* and *g½b* to *d½b* (= 343:512).

We now come to the circular mirroring of the three-note chord within the diminished fifth, and choose for that purpose the chord 7:8:10 = *f#–ab–c*, or its inversion 4:5:7 = *ab–c–f#*'. The representation of the major third and the seventh in the sketch shows the course of the mirroring: *ab –c–f#* through *ab* to *ab -f–b½b*; through *fb* to *c½#–fb–bb*; through *bb* to *g# -e–bb*; through *g#* to *g#–c½b(b#)–f½#*; through *c½b* to *e½b–c½b–f#*; finally back through *f#* to *ab -c–f#*.

1) Fokker here refers to the series – which occur in the diagram above as a diagonal progression – from *f#½–a–c½b–d½#*; and *d½b–e½#-g–bb½*. – Tr.

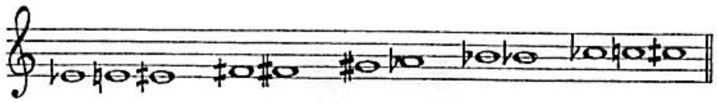
c

b½b ab f# e½b

fb c½b

c½# bb g# f½#

e



Again a *d* not notated is the center of the harmonic symmetry. The twelve-note scale thus created has seven dieses. Two of them have the relationship 125:128 namely *c–c½#* and *e½b –e*; the remainder all stand in the relationship 49:50. It is worthy of notice that *c–e* here is 3136:3125 larger than the pure third 5:4 There are thereby created a series of seven major thirds, from *c½#* rising up to *e½b*. *b½b* and *f½#* form a fifth in the relationship 1600:2401. Fifths in the relationship 512:343 lie between *e½b* and *b½b* and between *f½#* and *c½#*. The tone grouping contains further still two series of three successive sevenths and, on the diagonal, two series of successive tritones[1].

To conclude, we use the chord within the augmented fourth. We choose *f–ab–b* = 5:6:7. By grouping the notes in the sketch in minor thirds, tritones, and augmented seconds, the course of the mirroring is easy to follow: *f–ab –b* through *ab* to *b½#–ab–f½#*; through *b½#* to *b½#–d½#–f*; through *f* to *b–g#–f*; through *g#* to *f½b–g#–b½b*; through *f½b* to *f½b–d½b –b*; finally through *b* back to *f–ab–b*.

1) The intervals on the diagonal are less easily seen in the diagram. Here they are *c–f#–c½b–f½#* and *b½b-fb-bb–e*. – Tr.

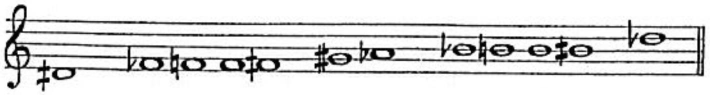
f

d½b b ab f½#

f½b b½#

b½b g# f d½#

b



Again the *d*, lying in the middle, but unexpressed, is the harmonic center of this group of notes. *b* and *f* appear twice; these twins vary from one another 125:126. The dïesis 49:50 is encountered five times, for example between *g#* and *ab*. The minor thirds *d½#* to *f½#* and *b½b* to *d½b* have the relationship 21:25. One finds further: two series of four successive minor thirds; two series of three successive tritones, at a distance of a dïesis of 35:36; finally in the horizontal rows two series of augmented seconds, at a distance of a minor third 21:25.

The Primary Chords with Four Notes

The conventional triad becomes a primary chord of four notes when the harmonic seventh is added to it. If the seventh above is added to a major triad, there results an (over) harmonic primary chord of four notes. If to the minor triad a seventh below (of the guidetone) is added, a sub-harmonic primary chord of four notes results.

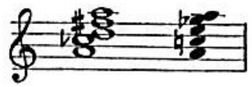
Furthermore one can combine several primary chords of four notes into a greater structure. It does not require much explanation that, by analogy to the connection of tonic, dominant, and sub-dominant, one can combine three primary chords of four notes. Their intervals need not be fifths or fourths; one can also utilize major thirds (or minor sixths) for this purpose, or pure sevenths, or supra seconds. The same is true of the subharmonic chords of four notes. In this manner one establishes connections of chords of four notes of similar sort, of *co-harmonic* chords of four notes.

The Mirroring of Primary Chords of Four Notes

We now come to tonal groupings of contra-harmonic chords of four notes, as they arise through mirroring (inversion). We complete first a chord of four notes with the octave of one of its notes, and mirror it within the framework of this octave. According to which note is doubled there are various relationships.

a) Mirroring at the fundamental or, respectively, the guidetone: *d–f#–a–c½b d'* becomes *d'–bb–g–e½#–d.*

b) Mirroring at the fifth: *a–c½b–d'–f#–a'* becomes *a'–gb'–e'–c'–a.*



c) Mirroring at the major third: *f#–a–c½b–d'–f#'* becomes *f#'–d#'–c'–a#–f*

d) Mirroring at the seventh: *c½b–d'–f#'–a'–c½b'* becomes *c½b'–a½#'–f½#'–d½#'–c½b.*

One could also say that the four-note chord is mirrored at the doubled not·

Further, one can choose two notes which are mirrored in one another whe· the four-note chord is inverted. From the four notes 4, 5, 6, 7, one can create six pairs: (4,5), (4,6), (4,7), (5,6), (5,7), and (6,7). The following examples result:

e) Mirroring with fixed 4,5: *d'–f♯'–a'–c½b'* becomes *f♯'–d'–b –g♯½*.

f) Mirroring with fixed 4,6: *d'–f♯'–a'–c½b'* becomes *a'–f'–d'–b½♯*.

g) Mirroring with fixed 4,7: *d–f♯–a–c½b* becomes *b♯ –g♯–e♯–d*.

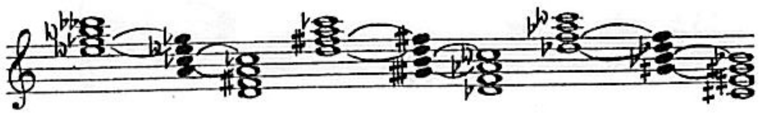
h) Mirroring with fixed 5,6: *d–f♯–a–c½b* becomes *c♯'–a–f♯–d♯½*.

i) Mirroring with fixed 5,7: *d–f♯–a–c½b* becomes *e½b'–c½b–a½b–f♯*.

j) Mirroring with fixed 6,7: *d–f♯–a–c½b* becomes *g½b'–eb½'–c½b–a*.

The Construction of Sequences Through Alternating Inversions

From the numbers 4, 5, 6, 7 one constructs two pairs. These can occur in three ways: (4,5) (6,7); (5,6) (4,7); and (5,7) (4,6). Now in the process of mirroring one tonal pair will be kept fixed, the other mirrored. The four-note chords of the pairs (4,5) (6,7) are, after two mirrorings, displaced by the amount of an octave and a small half-step.



Brought into the same octave, the four-note chords created through the alternating inversion construct a chromatic sequence. The half-step interval between the chords has the relationship 20:21, which corresponds to two dïeses.

If one takes the pairs (5,6) and (4,7) as the center of the inversions, one constructs a sequence which rises or falls in large half-steps. This large half-step has the relationship 14:15, which corresponds to three dïeses.

Finally if the pairs (5,7) and (4,6) are made the center of the inversions, the three chords stand in the relationship 24:35 to one another. The distances *d–a½b* and *a½b–eb* encompass seventeen dïeses.

Combined with the octave transpositions, the four inversions result in a raising of the pitch of a large half-step, which is computed as follows

$$\frac{1}{2} \times (35/24)^2 = 1225/1152$$

is only a little larger than 1224/1152 = 17/16, which corresponds to three di-
es.

Circular Mirroring of Primary Chords with Four Notes

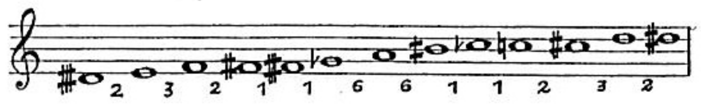
There are six number pairs available, through which the mirrorings of chords
: four notes are defined. If one chooses three pairs of numbers, one can carry
ıe circular mirroring through a cyclic use of inversion. If one makes the pairs
ı,6), (4,5) and (4,7) the center of the inversions, one creates three harmonic
ıords of four notes (with empty noteheads below) and three sub-harmonic
ıords of four notes (filled-in noteheads below).

'he resultant notes can be put together into a 12-note scale.

\ glance at the distance of the dieses in this scale shows that the series is sym-
netrical. The center of the symmetry obviously lies between *d* and *d*♯ on *d*½♯.
n two places three single dieses follow one another. It can be shown that ano-
:her series of the same pairs of numbers would lead to a transposition of this
series.

If the pairs (5,6) (5,7) and (6,7) are chosen, the following chords with four
notes result:

Again we place the notes together in a scale. This time there are 13 notes. Th
a turns out to be a constant.



In the given distances of the dieses, one notes that this series is also symmetri
Its center of symmetry lies on *a*. Nevertheless this *a* is not represented in any
the four-note chords by the numeral 4, which generally represents the funda-
mental or the guidetone. Again it can be shown that another series of the san
pairs of numbers would create a transposed but otherwise identical scale.

There are in all twenty various groupings of three pairs possible. We let the
two examples of its use suffice.

Eightfold Doubly-Linked Circular Mirrorings (Problème des Ménages)

Above we met a sixfold circular mirroring in which a single note (the *a*) re
mained constant. More closely considered, this note has in the chords of fou
notes, one after another, the function of the fifth above the fundamental (+6
6th overtone), the third below the guidetone (–5 = 5th undertone), of the se-
venth above (+7), of the fifth below (–6), of the third above (+5), and the se
venth below (–7). We summarize this briefly together in the formula (+6, –5,
+7, –6, +5, –7). The functions of the fundamental (+4) and of the guidetone
(–4) are not among these. Are there also circular mirrorings in which in eight
various chords of four notes, four harmonic and four sub-harmonic, one and
the same note experiences all eight possible functions? Yes, in abundance. C
need only choose the functions which the note concerned in the chord shou
fulfill in whatever series one after another with an alternation of plus and mi
signs: (+4, –7, +6, –5, +5, –6, +7, –4).

One sees that in this case not every chord of four notes has two notes in con
mon with its neighbor. Where the functions –5, +5 and –4, +4, follow one ar
ther, there is only *one* common tone with precisely the functions plus or mi
5 or plus or minus 4 respectively.

If one makes it a requirement that every chord of four notes should have ıubled connections with its two neighbors, the number of various possibilities comes much smaller. In the formula for its expression, then, neighbors may ɔt have the same number. This can be expressed in a graphic way. We want to troduce a four-note chord in which the note concerned has the function +4, us is the fundamental, as a gentleman with the name Four; we will call him ʌr. Four." His wife "Mrs. Four" would be the chord in which the chosen note ıs the function –4, thus it is the guidetone. Mr. Five would be the name of the ıord in which the chosen note has the function +5, thus is the third of the ındamental. Mrs. Six would be the name of the chord in which the chosen ɔte is the fifth below the guidetone, and so forth.

The requirement stated above corresponds to the following problem: for ɔur married couples who wish to find a place at a round table, a table order ıust be found according to which no man can sit next to his wife. Two table rders, which are constructed in mirror fashion as reading from left or right, we ·ould consider as alike. For three married couples there is an intelligible solu-on: every husband shall sit opposite his own wife, between the two other wi-ɛs. For four, five or indeed more couples, it is always more difficult to define ow many solutions there are. In mathematics this problem is well known. It arries the name of *Le problème des ménages*. The best minds have concerned ɪemselves to find a solution for whatever number of couples.

For four couples there are six orders at the table. We will seek to describe hem empirically. First we will seat the four gentlemen at the table. That gives ɩs the possibilities

+4 +5 +6 +7 +4

+4 +6 +7 +5 +4

+4 +7 +5 +6 +4

f one should change the places of two gentlemen, one gets the retrograde se-ʿies, the right-left mirror order, for example:

+4 +5 +7 +6 +4

ɩs the mirror image of

+4 +6 +7 +5 +4

With the gentlemen, one need only be concerned with the three orders above.

We now move on to the problem of placing the ladies. Mrs. Four, thus the number –4, must sit only between Messrs Five and Six or between Messrs Six and Seven, not however next to Mr. Four. If we seat Mrs. Four between Messrs

Five and Six, Mrs. Five can sit only between Messrs Six and Seven. If she shoulc not take this seat, then Mrs. Six or Mrs. Seven must sit there, which conflicts with the rule. Mrs. Five can thus only sit between Six and Seven. Thereupon follows the order

(1) +4 –7 +5 –4 +6 –5 +7 –6 +4.

Mrs. Four can also take her place between Messrs Six and Seven. Mrs. Five can then only sit between Messrs Seven and Four. Mrs. Six would then sit between Messrs Four and Five, Mrs. Seven between Five and Six. The formula now runs:

(2) +4 –6 +5 –7 +6 –4 +7 –5 +5

In similar fashion the order of seating can be ascertained if the gentlemen sit in the following series:

+4 +6 +7 +5 +4.

First Mrs. Four between Messrs Six and Seven; Mrs. Five only between Four and Six, Mrs. Seven only between Five and Four, Mrs. Six between Seven and Five. That results in this order of seating:

(3) +4 –5 +6 –4 +7 –6 +5 –7 +4.

Should Mrs. Four sit between Seven and Five, it necessarily follows that Mrs. Six sits between Five and Four, Mrs. Seven between Four and Six, and Mrs. Five between Six and Seven. As seating order, the result will be

(4) +4 –7 +6 –5 +7 –4 +5 –6 +4.

If the gentlemen take their places in the following series

+4 +7 +5 +6 +4,

the order at the table would be

(5) +4 –6 +7 –4 +5 –7 +6 –5 +4

(6) +4 –5 +7 –6 +5 –4 +6 –7 +4.

We translate all six table orders into tonal language and find six closed circular mirrorings for each of the eight primary chords of four notes.

The six rows of chords contain the same chords of four notes, only perhaps in another order. All have thus the same supply of notes, namely *thirteen* notes. If one looks at the intervals in the scale built from them, he finds exactly the same intervals.

6 1 1 2 3 2 1 2 3 2 1 1 6 dieses,

which we found above in a similar case with the fixed central note (there *a*, here *d*).

Combination Chords

If one lets two notes sound sufficiently loudly, the ear sometimes hears a third note, perhaps even a fourth one. The third note, TARTINI's *terzo suono*, has a frequency that is determined by the frequency difference of the sounding notes. One calls it therefore the "difference tone" and counts it among the combination tones to which also belongs the "summation tone," whose fre-

quency corresponds to the sum of the sounding frequencies. The combination tones are not „objectively" given, but are constructed only in the human ear. One can strengthen their effect through the device of sounding them objectivel and adding them to the given notes.

Frequency relationship such as 1:2:3 or 2:3:5 or 3:5:8 can be reproduced ir the conventional system of notation and on the usual keyboard instrument. Tl continuation of such series was previously not possible. Chords of three notes such as 5:8:13, 8:13:21 and so on cannot be illustrated with contemporary mu sical means.

Addition-series of this sort are known under the name of FIBONACCI. There are many such series. To them belong, for example, the series

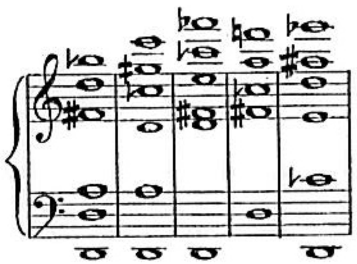
1 : 3 : 4 : 7 : 11 : 18 . . .

1 : 4 : 5 : 9 : 14 : 23 : 37 . . .

2 : 5 : 7 : 12 : 19 : 31 . . .

2 : 7 : 9 : 16 : 25 : 41 . . .

With the help of diesis notation these series can be represented with considerable accuracy[1].



Chains of Chords with Common Notes

We now come to chains of chords from addition-chords which have a common note. The first example shows the displacement within the overtone sounds over the constant *D*.

1) The first example, of course, represents the series 1:2:3:5:8:13. The others correspon to the table. – Tr.

The three moving voices rise in the manner of a progression in whole numbers (7:8:9:10:11:12 . . . respectively 9:10:11:12 . . ., respectively 5:6:7:8:9 . . .). In the course of this, the interval between the two upper voices is increased in size from 9:14 (*e–c*½♭) to 5:9 (*f*♯'–*e*'').

In the second example the *e* (= 36) of the upper voice remains in position and the second highest voice rises. That brings about a descent of the second voice from the bottom and the rising of the bottom voice, which merges into the *a* of the falling voice.

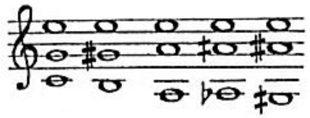
A fixed position of the highest note leads to a contrary motion of the two lower voices. A fixed position of the lowest note leads to a parallel motion of the two upper voices, convergent in their rising, divergent in their fall. The next two examples show a constant *f*♯' (= 20) with 20 = 8 + 12 = 7 + 13 = 6 + 14 = 5 + 15 = 4 + 16 and a constant *d*'' (= 32) with 8 + 24 = 9 + 23 = 10 + 22 = 11 + 21 = 12 + 20 = 13 + 19 = 14 + 18 = 15 + 17 = 16 + 16, with 21 = 10 + 11, 20 = 12 + 8, 19 = 13 + 6, 18 = 14 + 4 and 17 = 15 + 2.

The small adjustments in successive addition-chords of three notes with a constant note are, considered from the point of view of whole numbers, obviously alike. Measured in intervals they are however in inverse relationship to the numbers by which the notes are reckoned.

As a further example we take the chord 2:3:5 = *c–g–e*'. The *e*' will remain constant. The *g* becomes displaced by two dïeses downward, the *c* by three dïeses upward. The resultant chord, *d*♭*–g*♭*–e* = 3:4:7 is again a chord of addition. The *g*♭ is now lowered by three dïeses, the *d*♭ raised by four dïeses, in conformity with the inverted numerical relationship of these notes. *d*♯*–f–e*' with 7:8:15 is again a chord of addition. The next step brings then the octave.

5	7	15	2
3	4	8	1
2	3	7	1

Let us now follow the transformation in the inverted direction. The *g* is raised by two dïeses, the *c*' is lowered by three dïeses. Again *b–g*♯*–e*' = 3:5:8 is an addition-chord. Within the relationship 3:5, let the *g*♯ be raised by three dïeses, but the *b* be lowered by five dïeses. We arrive at *A–a–e*' = 1:2:3, with 1 + 2 = 3. One now adjusts the notes *a* and *A* by the amount of one and two dïeses respectively and gets the addition-chord *A*♭ *–a*½♯*–e* with 7 + 15 = 22. The *a*½♯ one raises by 7/7 of a dïesis, the *A*♭ one lowers by 15/7 dïeses and gets *G*½♯*–a*♯*–e*' with 7 + 16 = 23. One cannot pursue this process further because the dïesis notation is too gross for it. Only *G–a*♯*–e*' = 3:7:10 could be considered.



5	8	3	22	23
3	5	2	15	16
2	3	1	7	7

The notated chords of three notes belong to various fundamentals, respectively *c, e, a, b*½♭, and *a*♯. By means of transformation of three note addition-chords there is thus created a powerful mechanism for modulation.

The Simplest Three-Note Addition-Chords

From the abundance of three-note addition-chords, we select some particularly simple ones. We choose

2:3:5, for example *d–a–f*♯'

3:5:8, for example *a–f*♯'–*d*''

3:4:7, for example *a–d'–b*♯'

5:7:12, for example *f*♯–*b*♯–*a'*.

In special cases we might use the three-note chords

5:9:14, for example *f*♯–*e'–b*♯' and

9:16:25, for example *e–d'–a*♯'

whose intervals are respectively about a comma and a third of a comma different from one another.

Sequences of Linked Three-Note Addition-Chords

From the four especially simple three-note addition-chords one can construct six pairs. Each of these pairs lends itself to sequential progression, and each in three different ways. One might combine first 2:3:5 with 3:5:8.

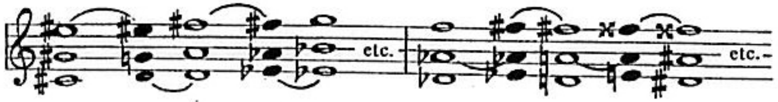
The *c* of *c–g–e'* = 2:3:5 remains and becomes the 3 of 3:5:8. Next the *a* (= 5) remains fixed and becomes the 3 of 2:3:5 = *d–a–f*♯'. If this operation is repeated several times, the chords are displaced at any one time by a major second: 9:10, thus five dĭeses. 2:3:5 = 18:27:45 is changed to 18:30:48 = 3:5:8 and again to 20:30:50 = 2:3:5. The upper voice goes from 45 through 48 to 50 traversing thus the half-steps 15:16 (= 3 dĭeses) and 24:25 (= 2 dĭeses).

Alternating, one can place the linkage in the lower and upper voices and can progress from 2:3:5 = 30:45:75 through 30:50:80 = 3:5:8 to 32:48:80 = 2:3:5. With each second chord the upper voice rises by about the minor second 15:16 (= three dĭeses). The middle voice alternately rises about five dĭeses and falls about two dĭeses.

A third possibility would be to place the connections in the upper and middle voices and from 2:3:5 = 48:72:120 progress through 45:75:120 = 3:5:8 to 50:75:125 = 2:3:5. In every second chord the upper voice rises about an infra second 120:125 = 24:25. The lower voice alternately falls by a minor second and rises by a major second.

The combination of 2:3:5 with 3:4:7 can equally be carried out in three manners. With the connections in the lower and middle voices the chords progress from 16:24:40 through 18:24:42 to 18:27:45. After every second chord, thus after each double operation, the notes of the original chord are raised by a major second 16:18 = 8:9. In the upper voice this rising is divided into an infra second 40:42 = 20:21 and a minor second 42:45 = 14:15.

The connections can also be placed in the lower and top voices or in the middle and top voices



	210 : 210 : 225	200 : 210 : 210
with the numbers	126 : 120 : 135	120 : 120 : 126
	84 : 90 : 90	80 : 90 : 84

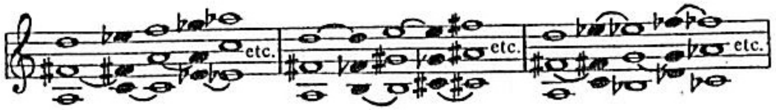
There follow the three forms of the combination of 2:3:5 with 5:7:12. The stepwise progression is performed in minor seconds (14:15 = 3 diēses), in infra seconds (24:25 = 2 diēses), and quarter-steps (35:36 = 1 diēsis).

70 : 72 : 75	120 : 120 : 125	175 : 180 : 180
42 : 42 : 45	72 : 70 : 75	105 : 105 : 108
28 : 30 : 30	48 : 50 : 50	70 : 75 : 72

We now combine 3:5:8 with 3:4:7. Major thirds (4:5 = 10 diēses), supra seconds (7:8 = 6 diēses) and mean seconds (32:35 = 4 diēses) arise in the melodic line.

28 : 32 : 35	49 : 56 : 56 : 64	256 : 280 : 280
16 : 20 : 20	28 : 35 : 32 : 40	160 : 160 : 175
12 : 12 : 15	21 : 21 : 24 : 24	96 : 120 : 105

With the combination of 3:5:8 with 5:7:12 we get, as characteristic intervals, a minor third (21:25 = 8 diēses), a major second (9:10 = 5 diēses) and a minor second (14:15 = 3 diēses).



168 : 180 : 200	144 : 144 : 160	112 : 120 : 120
105 : 105 : 125	90 : 84 : 100	70 : 70 : 75
63 : 75 : 75	54 : 60 : 60	42 : 50 : 45

Finally, the combinations of 3:4:7 with 5:7:12. The characteristic intervals here are an infra second (20:21 = 2 diēsis), a quarter-step (35:36 = 1 diēsis) an a fifth of a step (48:49 = 1 diēsis).

140 : 144 : 147	245 : 252 : 252	336 : 336 : 343
80 : 84 : 84	140 : 147 : 144	192 : 196 : 196
60 : 60 : 63	105 : 105 : 108	144 : 140 : 147

Isoharmonic Chords of Three Notes

The three intervals from which the major triad 4:5:6 is constructed create three difference tones, of which two coincide. The third of these sounds the octave above. Consequently the frequency of the middle note (5) is the arithmetic mean of the frequencies of the two outside notes (4 and 6). In the triad *d'–g'–b'* = 3:4:5 the difference tones of *d'–g'* and *g'–b'* = *G*. The difference tone of *d'–b'* lies an octave higher: *g*. The *G* here is the fundamental of the triad. The difference tone however does not always coincide with the fundamental. In the chord 14:19:24 = *f♯–c♭–e♭'* the difference tone, 5, is a *C*; the fundamental on the other hand is a very low *A♭*.

Chords of three notes in which one of the difference tones occurs doubled we will call isoharmonic. From such isoharmonic three-note chords, chains of chords can be constructed in which the intervals between the notes can be designated with the numbers 2, 3, 4, 5 and so forth.

8	9	10	11	12	13	14
6	7	8	9	10	11	12
4	5	6	7	8	9	10

12	13	14	15	16	17	18
9	10	11	12	13	14	15
6	7	8	9	10	11	12

16	17	18	19	20	21	22	23	24
12	13	14	15	16	17	18	19	20
8	9	10	11	12	13	14	15	16

20	21	22	23	24	25	26	27	28	29	30
15	16	17	18	19	20	21	22	23	24	25
10	11	12	13	14	15	16	17	18	19	20

In these examples the conventional three-note chords are particularly evident. (White notes in the example above. – Tr.) One sees that the new isoharmonic chords of three members subdivide the intervallic space of the conventional chords into 2, 3, 4, or 5 steps.

With two notes with the frequencies *m* and *n*, one can make the lower note (*m*) the difference tone and construct an isoharmonic chord of three notes from the *n*, (*n* + *m*) and (*n* + *2m*).

9	14	24	26	16	23
7	11	19	21	13	19
5	8	14	16	10	15

2	3	5	5	3	4

Here six examples are shown. It is evident that the diësis notation is not sufficiently precise for such chord structures. It simulates a pure major third for us where in reality there are intervals such as 24:19 and 26:21, which respectively are 96:95 larger and 104:105 smaller than 5:4. The chords 16:21:26 and 15:19:23 I have found in a new composition by Hans KOX.

Transposition by Means of Twofold Inversion

It has already been demonstrated that two notes of a harmonic interval, let us say 5:7, in addition to the common fundamental (1) also have a common guidetone (5 x 7 = 35). Should they stand with other notes over the same fundamental, for example with 4 and 6 over the fundamental 1, there is necessaril an inversion in which the 5 and the 7 remain unchanged in the new function as 1/7 and 1/5, and which permits the guidetone of the whole chord to remain 3 The mirrored center now lies about 1:35 higher than the original center. Correspondingly in the inversion of a subharmonic chord, with the retention of th notes 1/5 and 1/7, the center would be in the fundamental and would sink fro 35 to 1.

An example: we mirror the chord 4:5:6:7 by retaining the 5 and 6. The center changes position from 1 to 5 x 6 = 30. With the second inversion we let 1/4 and 1/7 remain. The center is changed now from 30 to 30/(4 x 7) = 15/14. Compared with the original chord the fundamental has been raised from 14 to 15, from *d* to *eb*.

In such a mirroring process octave notes such as 4 and 8 or 3 and 6 can also be retained. Then also the rule holds true that the distance between the fundamental and the guidetone corresponds to the product of the chosen numbers, here equals 4 x 8 or 3 x 6 respectively. In the event that in an inversion only *one* note remains invariant in the inversion, this note stands for two coinciding notes.

With these twofold inversions we are offered the possibility of carrying out in two steps any transposition we wish through an arbitrary number of diëses. The interval of 18 diëses can be expressed thus

$$\frac{6}{4} = \frac{4 \times 6}{4 \times 4} = \frac{6 \times 6}{6 \times 4} = \frac{5 \times 6}{5 \times 4} = \frac{7 \times 6}{7 \times 4}.$$

Correspondingly one can undertake four different transpositions:

The numbers set alongside the chords designate the notes of the chord which remain constant in the inversion. The interval of 12 diëses thus can be valued as

$$\frac{3 \times 7}{4 \times 4} \text{ or as } \frac{8 \times 8}{7 \times 7}.$$

That makes two transpositions possible.

The interval of 27 diëses can only be valued as $\frac{8 \times 8}{5 \times 7}$. There is therefore only *one* transposition possible:

We now assemble the possible values of the various diësis intervals in a table. I is sufficient to list only half of the diësis intervals, for the complementary int- val -- except the factor 2 (octave transposition) -- has the inverted value of th fractions.

1 Diësis:	$\frac{6 \times 6}{5 \times 7}$ $\frac{7 \times 7}{6 \times 8}$ $\frac{5 \times 10}{7 \times 7}$	30 Diëses:	$\frac{7 \times 7}{5 \times 5}$ $\frac{5 \times 7}{3 \times 6}$
2 Diëses:	$\frac{5 \times 5}{4 \times 6}$ $\frac{3 \times 7}{4 \times 5}$	29 Diëses:	$\frac{6 \times 8}{5 \times 5}$ $\frac{5 \times 8}{3 \times 7}$
3 Diëses:	$\frac{4 \times 4}{3 \times 5}$ $\frac{5 \times 6}{4 \times 7}$	28 Diëses:	$\frac{5 \times 6}{4 \times 4}$ $\frac{4 \times 7}{3 \times 5}$
4 Diëses:	$\frac{5 \times 7}{4 \times 8}$	27 Diëses:	$\frac{8 \times 8}{5 \times 7}$
5 Diëses:	$\frac{3 \times 3}{2 \times 4}$ $\frac{4 \times 5}{3 \times 6}$ $\frac{4 \times 7}{5 \times 5}$	26 Diëses:	$\frac{4 \times 4}{3 \times 3}$ $\frac{6 \times 6}{4 \times 5}$ $\frac{5 \times 10}{4 \times 7}$
6 Diëses:	$\frac{4 \times 6}{3 \times 7}$ $\frac{4 \times 8}{4 \times 7}$ $\frac{5 \times 8}{5 \times 7}$ $\frac{7 \times 8}{7 \times 7}$	25 Diëses:	$\frac{4 \times 7}{4 \times 4}$ $\frac{5 \times 7}{5 \times 4}$ $\frac{6 \times 7}{6 \times 4}$ $\frac{7 \times 7}{7 \times 4}$
7 Diëses:	$\frac{4 \times 7}{4 \times 6}$ $\frac{5 \times 7}{5 \times 6}$ $\frac{6 \times 7}{6 \times 6}$ $\frac{7 \times 7}{6 \times 7}$	24 Diëses:	$\frac{6 \times 6}{3 \times 7}$ $\frac{6 \times 8}{4 \times 7}$ $\frac{6 \times 10}{5 \times 7}$ $\frac{12 \times 7}{7 \times 7}$
8 Diëses:	$\frac{4 \times 6}{4 \times 5}$ $\frac{5 \times 6}{5 \times 5}$ $\frac{7 \times 6}{7 \times 5}$ $\frac{5 \times 10}{6 \times 7}$	23 Diëses:	$\frac{4 \times 5}{4 \times 3}$ $\frac{5 \times 5}{5 \times 3}$ $\frac{7 \times 5}{7 \times 3}$ $\frac{6 \times 7}{5 \times 5}$
9 Diëses:	$\frac{7 \times 7}{5 \times 8}$ $\frac{6 \times 10}{7 \times 7}$	22 Diëses:	$\frac{7 \times 7}{5 \times 6}$ $\frac{8 \times 10}{7 \times 7}$
10 Diëses:	$\frac{3 \times 5}{3 \times 4}$ $\frac{4 \times 5}{4 \times 4}$ $\frac{5 \times 5}{5 \times 4}$ $\frac{7 \times 5}{7 \times 4}$	21 Diëses:	$\frac{4 \times 6}{3 \times 5}$ $\frac{4 \times 8}{4 \times 5}$ $\frac{5 \times 8}{5 \times 5}$ $\frac{7 \times 8}{7 \times 5}$
11 Diëses:	$\frac{6 \times 6}{4 \times 7}$ $\frac{4 \times 8}{5 \times 5}$ $\frac{5 \times 9}{5 \times 7}$ $\frac{7 \times 9}{7 \times 7}$	20 Diëses:	$\frac{5 \times 5}{4 \times 4}$ $\frac{7 \times 8}{6 \times 6}$ $\frac{10 \times 7}{5 \times 9}$ $\frac{7 \times 14}{7 \times 9}$
12 Diëses:	$\frac{3 \times 7}{4 \times 4}$ $\frac{8 \times 8}{7 \times 7}$	19 Diëses:	$\frac{7 \times 7}{4 \times 8}$ $\frac{4 \times 8}{3 \times 7}$
13 Diëses:	$\frac{2 \times 4}{2 \times 3}$ $\frac{4 \times 3}{3 \times 3}$ $\frac{4 \times 5}{3 \times 5}$ $\frac{7 \times 8}{7 \times 6}$	18 Diëses:	$\frac{7 \times 6}{7 \times 4}$ $\frac{5 \times 6}{5 \times 4}$ $\frac{3 \times 3}{3 \times 2}$ $\frac{2 \times 3}{2 \times 2}$
14 Diëses:	$\frac{6 \times 8}{5 \times 7}$ $\frac{7 \times 7}{6 \times 6}$	17 Diëses:	$\frac{5 \times 7}{4 \times 6}$ $\frac{6 \times 12}{7 \times 7}$
15 Diëses:	$\frac{4 \times 7}{4 \times 5}$ $\frac{5 \times 7}{5 \times 5}$ $\frac{6 \times 7}{6 \times 5}$ $\frac{7 \times 7}{7 \times 5}$ $\frac{5 \times 5}{3 \times 6}$	16 Diëses:	$\frac{6 \times 6}{5 \times 5}$ $\frac{5 \times 10}{5 \times 7}$ $\frac{5 \times 8}{4 \times 7}$ $\frac{6 \times 10}{6 \times 7}$ $\frac{7 \times 10}{7 \times 7}$

Cadences

As is well known, many classical musical works close with a cadence which is constructed from the chords of the tonic, the sub-dominant and the dominant.

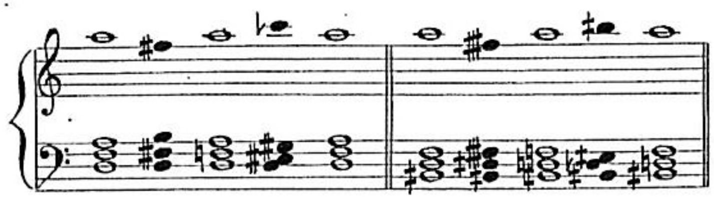
We now turn ourselves to the question of how, through a broadening of the concept of the classical cadence, other closings might be available. First of all, we can substitute the major third, the minor third or the supra second for the fourth occuring (in the bass) with the dominant and subdominant.

In these series of chords, various sorts of leading tones arise: minor seconds and infra seconds. If, with the help of the seventh, one makes the triads into primary chords of four notes, one gets here a still larger variety of leading tones.

To the harmonic cadences one can, mirror fashion, set up similar sub-harmonic closes. What was said of the leading tone is also true for these groups of chords.

A second sort of extension of cadences is created if one substitutes for the tonic, subdominant and dominant those chords which are linked together with a common tone.

With the next series of chords the bass notes form sub-harmonic triads (minc chords) for themselves; in this example the triads are 1/3:1/4:1/5, 1/4:1/5:1/6, 1/5:1/6:1/7, and 1/6:1/7:1/8. If these cadences are mirrored, guidetones arise which together with them form major chords.



The number of examples could be considerably increased.

A third sort of cadence construction would consist in that the bass notes be the difference tones from isoharmonic three-note chords. Naturally these three-note chords would generally not be ordinary major or minor triads. If once the difference tone is determined, we may at will designate one of the three notes of the chord as the sustained note; the other two notes of the three-note chord are thereby similarly fixed. The following examples will show how to proceed here.

The bass should proceed in equal intervals, in fourths, major thirds, minor thirds or supra seconds; evidently equal steps also lie in one of the upper voices, which we have chosen arbitrarily.

1)	9	13	14	9	30	19
	7	10	11	7	23	15
	5	7	8	5	16	11

2	3	3	2	7	4

1) Perhaps it is worth pointing out that Fokker is not concerned here with part-writing, but with the harmonic structure. – Tr.

9	24	17	9	19	30
7	19	13	7	15	23
5	14	9	5	11	16

2	5	4	2	4	7

The numbers set beneath the bass notes are related to the fundamentals of the chords in the upper staff. The examples above arise from *c* and *e'*, whose frequencies stand in the relationship 2:5. The first isoharmonic three-note chord to arise is the chord 5:7:9. With the second chord of the first example, *f* and *g*♯ stand in the relationship 3:7; thus the isoharmonic chord 7:10:13 is formed. In third place, the first chord returns. With the fourth chord, *G* and *c'* stand in relationship 3:8, through which the isoharmonic chord 8:11:14 is created.

To define the relationship of the *g*♯' to *e'*, one should note that *g*♯' has 7/3 the frequency of the *f*, and *f* again has 4/3 the frequency of *c*. The relationship of the frequencies *g*♯' to *c* is therefore 7/3 x 4/3 = 28/9. *e* and *c* stand in the relationship 5/2. The relationship of *g*♯' to *e* is therefore 28/9 : 5/2 = 56/45. It is smaller by 7.7 cents than the pure major third 55/44 = 5/4. In similar manner the frequency relationships in the following paragraph are derived.

The first example has fourths in the bass. Above it are major thirds: *g*♯, *e*, *c*. The interval of the third *e–c* is a pure third 5/4; the third *g*♯*–e* is somewhat too small: 56:45 = (55 + 1) : (44 + 1). The middle voice seemingly shows supra thirds, 9:7, whose true proportions are however *d'–b*♭½ = 80:63 = (9 x 9 – 1) : 9 x 7 and *b*♭½*–f*½♯= 14:11 = (11 x 9 – 1) : 11 x 7.

The second example has major thirds in the bass, supra seconds and infra thirds in the upper voices. The supra second *g*♭*–e* stands in the correct numerical relationship 8:7; *e–d*½♭ is 25:22 = 200:176 = 25 x 8 : (25 x 7 + 1). Similarly with the infra thirds: *b*♭½*–g* is 7:6, while *d*♭♭ -*b*♭½ is 115:98 = (119 – 4) : (102 – 4) = (17 x 7 – 4) : (17 x 6 – 4).

The third example has minor thirds in the bass. In the upper voices only one of the fourths is pure, the fourth *a–e* on the contrary stands in the relationship 168:125 = 8 x 21 : (6 x 21 – 1).

The fourth example has supra seconds in the bass. In the upper staff the *e–c* is a pure third 5:4, *g*♯–*e* on the other hand is reckoned 44:35 = (9 x 5 – 1) : (9 x 4 – 1). The middle voice runs through the thirds *d*♭'–*b*♭½ = 60:49 = 120:98 – (11 x 11 – 1) : (11 x 9 – 1) and *b*♭½–g♭ = 28:23 = (5 x 11 + 1) : (5 x 9 + 1). They vary only insignificantly from the relationship 11:9.

The cadential series mentioned last do not allow mirroring. With mirroring, the difference tones are altered. Through mirroring, three-note isoharmonic chords would lose their musical sense.

Table of English Equivalents for German Pitch Names

1.	c	c	19.	g	g
2.	ci	c semi-sharp	20.	gi	g semi-sharp
3.	cis	c sharp	21.	gis	g sharp
4.	des	d flat	22.	as	a flat
5.	dèh	d semi-flat	23.	àh	a semi-flat
6.	d	d	24.	a	a
7.	di	d semi-sharp	25.	ai	a semi-sharp
8.	dis	d sharp	26.	ais	a sharp
9.	es	e flat	27.	b	b flat
10.	èh	e semi-flat	28.	hèh	b semi-flat
11.	e	e	29.	h	b
12.	eï	e semi-sharp	30.	hi	b semi-sharp
13.	fèh	f semi-flat	31.	cèh	c semi-flat
14.	f	f			
15.	fi	f semi-sharp			
16.	fis	f sharp			
17.	ges	g flat			
18.	gèh	g semi-flat			

TABLE OF DIĔSES

The 31-note tempered scale and its approximations to the values of pure tuning

Diĕses	in decimal values	in cents	approximates	in cents
0	1,00 000	0,00	1/1	0,00
1	1,02 261	38,71	81/80	21,51
			64/63	27,26
			49/48	35,70
			45/44	38,91
			36/35	48,77
2	1,04 573	77,42	25/24	70,67
			24/23	73,62
			23/22	76,91
			22/21	80,53
			21/20	84,47
			20/19	88,80
3	1,06 398	116,13	16/15	111,73
			15/14	119,45
4	1,09 355	154,84	12/11	150,64
			35/32	155,14
			11/10	160,00
5	1,11 828	193,55	10/9	182,40
			28/25	196,20
			9/8	203,91
6	1,14 356	232,26	8/7	231,17
7	1,16 942	270,97	7/6	266,87
8	1,19 586	309,68	32/27	294,13
			19/16	297,51
			25/21	302,83
			6/5	315,64
9	1,22 290	348,39	11/9	347,41
			16/13	359,47
10	1,25 055	387,10	5/4	386,31
11	1,27 882	425,81	14/11	417,51
			32/25	427,37
			9/7	435,08
12	1,30 774	464,52	13/10	454,19
			64/49	462,35
			21/16	470,78
13	1,33 731	503,23	4/3	498,04
14	1,36 754	541,94	15/11	536,95
			11/8	551,32
15	1,39 846	580,65	7/5	582,51
			45/32	590,22

TABLE OF DIĔSES

The 31-note tempered scale and its approximations to the values of pure tuning

Diĕses	in decimal values	in cents	approximates	in cents
31	2,00 000	1200	2/1	1200
30	1,95 579	1161,29	63/32	1172,74
			96/49	1164,30
			88/45	1161,20
			35/18	1151,23
29	1,91 254	1122,59	23/12	1126,38
			21/11	1119,47
			19/10	1111,20
28	1,87 026	1083,88	15/8	1088,27
			28/15	1080,55
27	1,82 890	1045,17	11/6	1049,36
			64/35	1044,86
			20/11	1040,00
26	1,78 847	1006,46	9/5	1017,60
			25/14	1003,80
			16/9	996,09
25	1,78 846	967,75	7/4	968,83
24	1,71 025	929,04	12/7	933,13
23	1,67 244	890,33	27/16	905,87
			5/3	884,36
22	1,63 546	851,62	18/11	852,59
			13/8	840,53
21	1,59 930	812,91	8/5	813,69
20	1,56 394	774,20	11/7	782,49
			25/16	772,63
			14/9	764,92
19	1,52 936	735,49	20/13	745,81
			49/32	737,65
			32/21	729,22
18	1,49 555	696,78	3/2	701,96
17	1,46 242	658,07	22/15	663,05
			16/11	648,68
16	1,43 008	619,36	36/25	631,28
			10/7	617,49

MUSICAL WORKS IN THE 31-NOTE SYSTEM

HENK BADINGS

Preludium en fuga voor 31-toonsorgel	1952
Contrasten (H. De Vries) voor gemengd koor a cappella	1952
Preludium en fuga IV voor 31-toonsorgel (III et II desunt)	1954
Suite van kleine stukken voor 31-toonsorgel	1954
Reeks van kleine klankstukken in selectieve toonsystemen, voor 31-toonsorgel	1957
Sonata no 2 for two violins	1963
Sonata no 3 for two violins	1967
String Quartet	1967
Concerto for two violins and orchestra in 31-tone temperament	1969

ANTON DE BEER

Method for the 31-note keyboard	1961
Instructive Sonatina	1964
Small pieces	1967
"Speelmusiek" for two violins and archiphone	1971
Music for the archiphone	1973

JAN VAN DIJK

Vier harmonisch-melodische intonatie-oefeningen voor strijkkwartet	1946
Acht stukken in elementaire geslachten van Euler (orgel)	1948
Vierstemmige intonatie-oefeningen voor gemengd koor a cappella, bij een studieboek van prof. Fokker	1949
Musica per organo trentunisono I	
I. Quattro pezzi per organo trentunisono solo	
II. Quattro pezzi per organo e archi	
III. Dieci pezzi per organo e strumenti diversi	
IV. Concerto per organo e orchestra	
V. Canzone in genere enharmonico vocale per organo e canto ad lib.	1950/51

Deuntje (J. Van den Vondel) voor dubbel gemengd koor a cappella	195
Hymne (P. Minderaa) voor gemengd koor a cappella	195
Musica per organo trentunisono II	
Sette pezzi per organo solo	195
Concerto per trombone, violino e violoncello	196

EUGEN FRISCHKNECHT

Drei Stücke für 31-Ton-Orgel	196

JAAP GERAEDTS

Zes studies in Euler's toongeslachten voor fluit, en voor twee fluiten	1961/6

ANTHON VAN DER HORST

Suite voor 31-toonsorgel, op. 60	19:

ARIE DE KLEIN[1]

Zes bagatellen op tien toetsen	
a) Genus chromaticum, b) Zee en haven, c) Stad en land, d) Chinees, e) Moed en angst, f) Twee koekoeken	1950/:
Preludium chromaticum	19:
Kalenderblaadjes in de kringloop van acht	19:
Passacaglia	19:
In generibus Leonhardi Euleri	
($3^2.5.7$) Toccata, ($3.5^2.7$) Barcarole e toccata, ($3.5.7^2$) Il sogno di Giacomo, ($5^3.7^2$) Mordente e toccata, ($5^2.7^3$) Mosaico e toccata	19:
Kringspiegelingen	19:

HANS KOX

Drie stukken voor viool solo, in ($3^3.7^2$)	19:
Vues des anges (R. Rilke), voor viool en bariton	19:
Passacaglia en koraal, voor 31-toonsorgel	19(
Vier stukken voor strijkkwartet, 31-toons	19(
Vier didaktische stukken, voor twee trompetten en een trombone	19(
Serenade for two violins	19(

1) I am indebted to Joel Mandelbaum for the information that Arie de Klein is a pseudony for Professor Fokker himself. – Tr.

TON DE LEEUW

Elektronische studie	1957

JOEL MANDELBAUM

Ten studies in 31-tone temperament for organ	1963
The Dybbuk, Opera (Act III)	1968
Three Songs for soprano, two violins and archiphone	1971

RICHARD ORTON

Mosaics, for 31-tone organ	1965

OEDOEN PARTOS

Three fantasies for two violins	1972

ALAN RIDOUT

Partita for cello solo	1959
Music for 31-tone organ	1960
Trio for strings, I. Chants, II. Dances, III. Variations	1961
Sonata for two violins	1965
Animula vagula blandula (Hadrian), for violin and baritone	1965

PETER SCHAT

Collages, voor 31-toonsorgel I. Contrastes, II. Canons, III. Clusters	1962

ALPHONSE STALLAERT

Vier liederen voor twee violen	1965

PAUL CHR. VAN WESTERING

Zes inventies voor 31-toonsorgel	1951
Variaties over "Merck toch, hoe sterck," voor 31-toonsorgel	1951

IVAN WYSCHNEGRADSKY

Étude ultrachromatique pour l'orgue trente-et-unisonique	1

WORKS BY ADRIAAN FOKKER WHICH DEAL WITH MUSIC

Les mathématiques et la musique, Den Haag (Mart. Nijhoff) 1941.

Rekenkundige bespiegeling der muziek, (Gorinchem (Noorduyn) 1945.

Just intonation, Den Haag (Mart. Nijhoff) 1949.

Recherches musicales théoriques et pratiques, Den Haag (Mart. Nijhoff) 195

Oor en stem, bundel solfège-oefeningen, Amsterdam (Alsbach en Co.) 1964.

Les gammes et le tempérament égal, in: Acustica 1, Zürich 1951, 29.

Equal temperament and the thirty-one-keyed organ, in: The Scientific Mont 81, 1955, 161.

Les cinquièmes de ton, in: *Acoustique musicale*, Paris (Ed. centre nat. reche che scientif.) 1959

Optelakkorden, in: Mens en melodie, Utrecht 1960, 145.

Warum und Wozu? in: Die Reihe 8, Wien 1962, 62.

Multiple antanairesis, in: Proc. Kon. Nederl. Akad. v. Wetensch. A 66, No. 1 1963.

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Band 6	Giuseppe Tartini, Traktat über die Musik gemäß der wahren Wissenschaft von der Harmonie, übersetzt und erläutert von Alfred Rubeli, 397 Seiten, Halbleinen DM 66,00
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