The Tonality of the Golden Section

by

Walter O'Connell

PREFACE

The mathematical ratio 1.618...:1, traditionally denoted as <u>phi</u> and variously known as the golden section, the golden mean, the divine proportion, has fascinated artists and architects since antiquity. Undoubtedly a number of attempts to exploit its musical potentials have arisen independently. One such possibility first occurred to the present author in 1960. As a matter of historical interest, the following previously unpublished paper is presented here in approximately its original form. For clarity, this preface describes the context in which it was written, while the afterword provides a brief update.

An article entitled 'Tone Spaces' was accepted by Karlheinz Stockhausen in 1959, and published in German translation in <u>die Reihe</u> vol 8 in 1962. In 1965 an expanded version was prepared for the English edition (Theodore Presser, 1968), with the following material to be added as a new Part III. However, space limitations prevented its inclusion. A few copies were circulated privately beginning in 1967.

Parts I and II of 'Tone Spaces' were concerned with new methods of transforming a given 12-note row to generate a number of variants. Unlike the sorts of permutation characteristic of much serial thinking at that time, these methods all stemmed from various <u>geometrical</u> representations of the row (as had the original inversion, retrograde, retrograde inversion): hence the term <u>tone spaces</u>. While such representations can faithfully model the group properties of the intervals (e.g., how two intervals can combine to form a third interval), they make almost no reference to the acoustic properties of tones (their overtones, combination tones, etc.) that underlie both traditional harmony and modern attempts to extend that domain. So while an exact inversion, e.g., may mirror the <u>shape</u> of the original idea, it may sometimes vitiate whatever harmonic drive was present, or render it 'nonsense.' A good 12-note row must be very carefully chosen with due regard for all the forms in which it is to appear.

It was in this context that the tonality of the golden section was offered as an <u>acoustically</u> more symmetrical material, hence one more amenable to serial treatment. However, it is now clear that this motivation is not essential. Analogs of tonality reappear, and may be accentuated or suppressed as the new domain is explored in many possible directions.

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THE SYMMETRIZATION OF TONE SPACE

Thus far we have been dealing with a kind of note space, although in the tone lattices our 'notes' did acquire an additional reality of being assigned a specific octave register. But we have seen that as yet so little reference has been made to the acoustic nature of our material that our methods would equally well apply to any variable having an appropriate equivalence relation (congruence modulo 12) that reduces it to a cyclic group of order 12. Let us now admit a bit more reality into our abstract representation.

We sound a tone on any orchestral instrument of definite pitch (with the exception of certain percussion instruments such as drums, gongs, bells). What physical signal is presented to our ear? A series of frequencies, f, $2f$, $3f$, $4f$, $5f$,...: the harmonic overtone series. Examining these partials in more detail, we find that they include:

(1) the sine tone of fundamental frequency f to which the note name is assigned;

(2) the first overtone $2f$ (2nd harmonic) at the interval of the octave, which we take to be the identity interval of our scale structure;

(3) the second overtone $3f$ (3rd harmonic) a fifth above $2f$, this fifth as we have seen being an interval upon which the twelvefold division of the octave is based; we may regard it as the generator of the twelve note scale;

(4) $4f$, the next octave, and $5f$ which completes the major chord formation together with $6f$;

(5) $7f$, a quite flat minor seventh unused in traditional western scales, together with higher overtones that become increasingly more crowded, filling the next octave with adjacent seconds (progressing from major to minor seconds to intervals still smaller in the following octave).[1]

Within this sound structure are suggestions that historically have been followed up in various systems, no one of which is uniquely determined by the sound material itself: organum, both Greek and Church modes, classical tonality, atonality, 12-tone and serial techniques. Throughout this development, as the implications of the items towards the beginning of this list (lower partials) were pressed more and more stringently towards their logical conclusions, the presence of all higher partials became increasingly obtrusive. The highest overtones cause no difficulty for unison voices. Yet even the third,
13 fourth, and fifth harmonics, so essential to the classical derivation of the major chord, mark an undesired usurping of musical space that can interfere with the twelve-tone ideal of clarity; in the pursuit of this ideal we have become increasingly sensitive to the masking of higher tones by the partials of lower tones.

Even the remarkable ability of this complex of sinusoids to fuse into a single perceived tone is the result of a feature at variance with our present symmetry requirements: the difference frequency formed by any two adjacent overtones is always the fundamental frequency. Whenever two tones are sounded simultaneously, the nonlinear response of the ear gives rise to various combination tones, the two most important of which have frequencies equal to the difference frequency, $f_2 - f_1$, and (with lower intensity), the summation frequency, $f_1 + f_2$. Now $5f - 4f = f$; and $9f - 8f = f$; etc. So we see that the 'fundamental' is well named as not only the lowest partial present, but as the one tone to which all the others lead. The concept of chord roots and thorough-bass procedures lean on this acoustic attribute of tone structure.[2] Thus there is a clearly defined asymmetry in our present tone space, itself an organizing principle which Hindemith has appropriately likened to gravity. Its presence together with the above factors belies any slavish adherence to techniques based exclusively on the group properties of note names; yet perhaps just this conflict between opposing tendencies latent within the material is essential to our present instrumental dialectic.

Nevertheless, if only as an intellectual exercise, we might consider whether a more fully symmetrical tone space is conceivable. Let us freely list some desirable properties for the structure of a tone closer to our ideal.

(1) The partials will occur only at the equivalence interval, so that it can mask no tone having a different note-name, usurping no tone space other than its own.

(2) The combination tones formed between adjacent partials will, to as great an extent as possible, duplicate other partials; thus the clarity achieved by requirement (1) will be preserved.

(3) It will have no 'fundamental,' no one tone to which all others point; it will have neither top nor bottom, but will extend by implication beyond the range of audibility in both directions.

(4) It will be a 'good Gestalt'---having a simplicity and interrelatedness of structure that will allow it to cohere as a perceptual unit when it is heard amidst other such tones.

Let us denote the new equivalence ratio as x. Then by (1) the ratio of the frequencies of two successive partials is $f_2/f_1 = x$. For our conventional scale, $x = 2$, the octave; this provides one possible solution for our requirements. In this case, (2) is in part satisfied, in that $2f - f = f$: the difference frequency for two adjacent partials is the lower of the two partials. (3) will be satisfied if we begin with a frequency f below the audible range, and proceed with a series of partials $2f$, $4f$, $8f$, $16f$,... out through the upper limits of audibility. (4) is largely satisfied by (1) and (2); however, the aural coherence is distinctly weaker than that of the harmonic overtone series where all integer multiples of f are present. Here, because of the peculiar way in which (2) is satisfied, each partial is linked only to its lower neighbor, in a chain-like fashion. Moreover, (2) is violated by the summation-tone, since $f + 2f = 3f$, not one of the partials defined by (1). However, the presence of $3f$, a fifth above $2f$, could define the generating interval of the scale; again we might approximate the Pythagorean derivation with the even-tempered twelve-tone scale. In this first case, what has resulted is a somewhat emaciated, but acoustically purified version of our traditional tonal material.

Throughout this section, *harmonic* is used in its technical mathematical sense: any one of a series of pure sine tones whose frequencies are integer multiples of a fundamental frequency (that of the first harmonic). *Overtones* may be either harmonic or non-harmonic, though there is still the suggestion of a lowest, or fundamental frequency. *Partial* is least specific, referring to any one of a complex of sine tones, however their frequencies may be related.

We do not seriously propose this octave overtone structure as a desirable tone formation. Rather, it is offered to illustrate the nature of the requirements (1-4) that we have imposed. We might call it the zeroth-order solution for those conditions. Before investigating other solutions, let us narrow the field further by adding one more condition.

(5) To as large an extent as possible, relations occurring within the tone will be symmetrical with respect to inversion. This is in fact the acoustic condition needed to support a serial treatment of tone rows with primary emphasis on their group properties: the symmetrization of tone space.

We see that the octave-overtone solution does not satisfy this condition, because of the complete asymmetry between summation tone and difference tone. Nevertheless, we have taken a first step by giving no acoustic preference to the major over the minor triad, for example; and the distinctions between chord positions (root, third, fifth, etc.) have disappeared with the removal of a unique fundamental, and the distribution of partials though all octaves.

Let us now seek a first order solution for all five conditions. We begin by requiring that the difference tone between two adjacent partials be the next lower partial, and that their summation tone be the next higher partial. Again designating the equivalence ratio as x, the ratio of frequencies of any two adjacent partials is equal to x. Designating the frequency of the lower of the two partials as f, the higher has a frequency xf. The difference tone must be the next lower partial, f/x; the summation tone the next higher, x^2f. Recalling their definitions, we may write:

DIFFERENCE TONE	SUMMATION TONE
$xf - f = f/x$	$x^2f = xf + f$
$x - 1 = 1/x$ or $x^2 - x = 1$, so $x^2 = x + 1$	$x^2 = x + 1$

Fortunately our two requirements each lead to the same mathematical condition, so it is possible to satisfy both requirements simultaneously and produce the symmetry we have demanded.[3] The solution is the famous ratio known to antiquity, and termed the golden section. When a line segment is subdivided into two sections such that the whole length is to the longer as the longer is to the shorter length, each of these ratios equals the golden section. Its many remarkable properties, together with
12 its recurrence throughout the art and architecture of many eras and its pervasive influence on many natural forms, have been extensively discussed elsewhere.[4] Wherever it has occurred it has played a

role analagous to our identity ratio--to such an extent that Stockhausen could say of the octave, "The dominating *2-relationship* seems to rest on a fundamental principle of our sense perception, to be the acoustical 'golden section.'"[5] We shall examine that proposition at the close of this section.[6] But first we consider its converse: the golden section will become our new 'octave,' taking us another step toward a unified esthetic.

THE TONALITY OF THE GOLDEN SECTION

The preceding equation leads to the frequency $x = (1 + \sqrt{5})/2 = 1.61803...$, the golden section, which is to be our new equivalence ratio, and hence the ratio between any two adjacent partials of the tone we are synthesizing. Expressing this musical interval in terms of the tempered 12-note scale, we find it to be a minor sixth plus a third of a semitone (8 1/3), with the remarkably high precision of 833.1 cents. A series of successively high partials spanning the entire audible range reads thus:

$$C, A^{b+}, F^{-}, C^{\#}, A^{+}, F^{\#-}, D, B^{b+}, G^{-}, E^{b}, B^{+}, A^{b-}, E, C^{+}, A^{-}, F, C^{\#+}$$

where + designates a third of a semitone above the tempered intonation and - a third of a semitone below the tempered note named. Here then is the structure of a single tone from which we propose to build a new tonality.

A scale compatible with such a tone structure is already strongly suggested by the above naming of partials in terms of sixth-tones of the tempered twelve-note scale. The new equivalence ratio encompasses just twenty-five such sixth-steps. But what logic lies behind the choice of such a scale? In the case of the 12-note scale, we regarded the (tempered) fifth--the interval between the second and third harmonics--as the generator of that scale. Is there an equally natural choice of generator for the newly proposed tone? Its very construction has been based upon one consequence of the ear's non-linear response: our ability to 'hear' combination tones even though they have no physical correlate in the sound waves impinging on our eardrums. Another consequence of that same non-linearity is the ear's creation of the first few harmonics for any pure sine tone supplied to it at sufficient intensity level. Thus, even though we have purged the physical stimulus of the last remnants of the familiar overtone series, the ghosts of these harmonic overtones return to haunt us. It is not merely that we remember them for their ubiquity in our sound world. (Admittedly, vibrating strings and air columns, and in fact all non-rigid one-dimensional oscillators will continue to produce them.) For we propose to move beyond that too-familiar world into a domain created electronically to our exacting specifications. Technologically, this is entirely feasible. Yet such sounds enter our consciousness through a still 'imperfect' instrument--an ear that produces harmonic distortion. From the standpoint of Fourier analysis, the only vibration having a single frequency is a pure sinusoid; any distortion of a sine tone introduces additional harmonics, the sinusoids (at integer multiples of the original frequency) needed to describe this distortion.[7] That we are capable of making an aural Fourier analysis is indicated by our ability to identify harmonics presented to the ear, though we retain some option about the degree and manner of analysis employed in our listening. More remarkably, experimental evidence suggests that the inner ear produces faint subharmonics as well. In any case, it would seem that we must accept the presence of the most prominent aural harmonics: the second harmonic (at the octave), and to the lesser extent, the third harmonic (at the twelfth). Thus we are led to choose the octave as the generating interval for our new scale: this choice will be found to be compatible with the faint presence of the third harmonic (now far less audible than for conventionally produced tones) which gave rise to the generating interval for the 12-note scale.

Our course is now clear. Just as the chromatic scale was derived by spelling out a succession of twelve ascending fifths, tempering the result so as to achieve closure (adjusting the size of the fifths so that the thirteenth tone was exactly seven octaves above the starting point), and then using the equivalence relation (octave) to bring all the resultant notes into the same register, here we proceed by exact analogy, starting with a succession of twenty-five ascending octaves, adjusting for closure, and using the golden section as the equivalence ratio to reduce these notes to the same register. Beginning from C, the first octave above is lowered to E-; the second octave to G+; the third octave must be lowered not to B, but to E^{b}- to bring it into the same register. Here we are seeing the new scale spelled out on 'the circle of octaves,' exactly analagous to the former circle of fifths. Just as the pentatonic and diatonic scales in their tempered intonation result from stopping the procedure short of closure, we might stop here with a scale of just nine notes. This scale consists of seven half-steps and two third-steps. While the two smaller intervals may be arranged in a manner reminiscent of the major scale, when we carry out the exact analog of the procedure for generating the major scale we obtain:[8]

$$C \quad C^{\#-} \quad D^{-} \quad E^{b-} \quad E^{-} \quad F^{-} \quad F^{+} \quad F^{\#+} \quad G^{+} \quad A^{b+}$$

2 3 3 3 3 2 3 3 3

The numbers listed below the note names indicate the interval sizes, measured in the scale-steps of the 25-note scale that results when the generating procedure is carried through to closure.

Within the new 25-note 'chromatic' scale, in addition to the modes of the nine-note scale, many other scales using less than the full complement of notes are possible. Like the various modalities excised from the complete twelve-note set, such scales will arise from giving precedence to certain intervals exemplifying a close 'harmonic' connection between notes, and accepting the presence of other distant relationships only to the extent of effecting a premature closure.[9] Not until the full 25-note set is accepted can the intervals again function with complete equality before the law. In approaching the new material, we have the option of recapitulating the historical developments from modality to tonality
13 and beyond, by establishing a hierarchy of interval relationships.

The frequency ratios, listed roughly in order of increasing complexity, are:

2:1	4:1	8:1	3:1	3:2	4:3	3:8	$\sqrt{5}$:1	$\sqrt{5}$:2	4:$\sqrt{5}$	8:$\sqrt{5}$	3$\sqrt{5}$:2	3$\sqrt{5}$:4
+11	−3	+8	+7	−4	−10	+1	−8	+6	+5	−9	−12	+2

(Beneath the frequency ratio is the reduced interval which it approximates in sixth-steps.) Other simple combinations not shown here lead to the same intervals or their inversions. (Note that $\sqrt{5}$ is the difference frequency between a partial and the octave of the next partial above it, divided by the frequency of the original partial.)

An harmonically constructed modality should then clearly include the +11 with high frequency of occurrence, together with 3's, 7's, and/or 8's, and possibly 4's. The sizes of the scale-steps, measured in sixth-tones, are listed for a few possibilities, arbitrarily arranged in their most symmetrical forms.

7-note scale: 4, 3, 4, 3, 4, 3, 4 | 11-note: 2, 3, 2, 2, 2, 3, 2, 2, 2, 3, 2

9-note scale: 3, 3, 2, 3, 3, 3, 2, 3, 3 | 14-note: 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2

The search for a medium more fully compatible with serial technique first led us to the tonality of the golden section, though the 'harmonic' properties remind us that again other forces are at work within it, now far more symmetrically distributed than heretofore. What are the group properties of modulus 25? Immediately we see that it has only one prime divisor, 5. Hence multiplication of all intervals of a row by a factor other than 5 produces a true interval transformation, having all the invariants exhibited by the previous [1, -5] transformation of *Tone Spaces I*. Since 25 is not one of the 'crystalline congruences' discussed in *Tone Spaces II*, no non-redunant all-interval set exists, and no natural extension to a three-dimensional lattice structure has been found.[10]

With the existence of abundant interval transformations, the primary problem seems to be that of producing coherence within a 25-note row, seemingly too long to produce a recognizable Gestalt. Here the fact that $25 = 5^2 = 3^2 + 4^2$ may aid us. If the scale is divided into five cells, each containing five adjacent scale tones numbered 1 through 5, a pentachord may be chosen in such a way that it draws a different note-number from each cell. For example, we might choose the fourth note of the first cell, the second note of the second cell, the fifth note of the third cell, the third note of the fourth cell, and the first note of the fifth cell. Any such choice produces a transposable pentachord: transposing the above five notes by +5, +10, +15, and +20 fills out the complete twenty-five note set without duplication. Such pentachords are analogous to the tranposable hexachords or triads from which Webern's rows are sometimes constructed. No exact analog exists for the invertible hexachords of Schoenberg's rows. However, a new symmetry has appeared: there are five sets of five notes each. This means that a cyclic permutation of order-numbers can be carried out within the pentachord as its five transpositions occur.[11] Moreover, the row may also be regarded as three triads plus four tetrads. If the notes of the triad are three adjacent tones, and those of the tetrad an arrangement of the next four tones, then transpositions at +7, +14 (and +21 for the triad) will spell out the 25 notes without duplication. All transformations of this set by interval multiplication will retain this transposability. Again, any permutation cycles can be brought to completion within the row, since we have just three transpositions of the three-note set, and four of the four-note set.

It would be premature to speculate further about compositional procedures for a tonality as yet unheard, since its conception in 1960. Let us merely add that the golden section is equally natural as a basis for duration ratios. Here a twenty-five-fold subdivision is entirely unnecessary; rhythmic structures arise from combinations of powers of 2, 3 and the golden section, with their differences generating other scale intervals. Throughout we have assumed a purely electronic realization. Yet we must not preclude the possibility of electronic instruments for performers who have added a feeling for the duration ratio 1.62:1 to their repertoire, or which have a kind of duration-keyboard, as Stockhausen has suggested. Steps toward this rhythmic conception have been taken by all composers who have utilized the Fibonacci series in duration structures.

Some acoustic properties of the new 25-note scale and its derivatives require further comment: the perceptibility of the small interval size; the effects of beats formed for the more 'dissonant' intervals. So far as the generating interval, the octave, is concerned, tempering requires an utterly imperceptible alteration of 0.350¢(36/25 of the difference between the golden section, 833.0903 cents, and its represention as 25 sixth-steps, or 833.3333 cents). Compared to the tempered fifth (the generating interval of the 12-note scale) which is 1.955 cents low, this is an absolute error only one sixth as great--or, considering the smaller interval size, it is still only one half as large. The cycle of 12 perfect fifths fails to close by 23.46 cents, while here the cycle of 25 perfect octaves fails by only 8.75 cents. Of course the fifth remains, but now as a secondary relation analogous to the major third of the 12-note scale; here again there is more than a 5-fold increase in accuracy, replacing the 13.7-cent error in the third by the 2.15-cent error in the fifth. Just as the seventh harmonic is excluded from representation in the tempered chromatic scale, so here the fifth harmonic is excluded. The tertiary relationships involving $\sqrt{5}$ are much more poorly approximated. However, remembering that we now are concerned only with subjectively produced harmonics, we find that on the whole the tempered 25-note scale represents the ideal of the golden section tonality with an astonishing accuracy, far exceeding that of even the 53-note tempered approximation of just intonation.[12]

The small interval size may seem problematical. The sixth-step itself could be avoided altogether within the confines of subsidiary scale formations (much as enharmonic changes are classically avoided). And just as there are quasi-diatonic 12-note rows, analogous formations emphasizing the harmonically strong relations are possible here. However, it is anticipated that even the smallest intervals will be clearly audible. In the intermediate range of frequencies, the sixth-step is far larger than the smallest
13 perceptible interval, though pitch discrimination decreases in the outer registers. The very structure of our tones, having partials of varying amplitude in all registers, assures that some partials will be present

in the range of optimum discrimination. Interval perception should be considerably enhanced by the absence of the crowded upper harmonics and the reiteration of the same interval by all partials. For the smallest intervals, the difference frequencies are of course perceived as beats, and it might seem that their frequencies would become so low as to become disturbing. But here again the tone structure is to our advantage: the beats between adjacent partials form a series of frequencies in exactly the same ratios as the partials themselves. Thus the aperiodic character of the individual tones carries over as an aperiodic pulsation imposed on 'dissonant' combination, relieving them of the mechanical effect associated with single beat frequencies.

How wide a variety of tone colors may be synthesized? No full answer can be given for the present. The entire range of string and wind sounds is achieved merely by altering the relative amplitudes of the various harmonics of the conventional overtone series. However, at this stage we deal with a somewhat depleted set of partials, roughly analagous to including only the octave overtones of the conventional series. Some considerable reduction in the range of timbres available might then be expected when we can vary only the relative intensities of this restricted set of partials. Yet the fact that they each correspond to a different note-name in a 36-fold subdivision of the octave may widen the range of qualities in an as yet unpredictable way. If it proves desirable to enrich the tone structure further, new partials (of variable amplitude) may be added at the difference and summation frequencies of already existing non-adjacent partials, as well as at the first four harmonics (and for symmetry, possibly the corresponding subharmonics), and at additional octaves above and below.[13]

By now it must be evident that the new tones have no assignable note-names of the sort we have used to designate their partials, nor can their 'register' be specifically indicated. Pitch numbers from one to twenty-five can serve as note names. Depending upon the row structure, pitches may conveniently be designated within a 5 x 5 array, or a 3 x 3 plus a 4 x 4 array. These are the note spaces natural to our new material. If the pattern of intensities of successive partials retains a more or less definite form, changes of register correspond to shifts of this pattern. When the pattern has a sharp single peak a definite register can be assigned, but patterns with multiple peaks, or even complete uniformity are possible. The new tones remain quite definite, but conventional designations fail to describe them.

Since we now deal with 12 interval-magnitudes, a 12-note row may be transcribed as a 12-interval-row in the new sound medium. And $12 = 3 + 4 + 5$, while $25 = 3^2 + 4^2$, so some triad, tetrad and pentad relations may be carried over from one note row to another. Moreover, under an interval-multiplication of 3, a 25-note row becomes expressible on conventional instruments. From the standpoint of the new tonality such an instrument plays *all* the notes, but in restricted registers.

With additional instruments, some tuned a sixth-step high, others a sixth-step low, the full complement of registers becomes available, incidentally allowing for perhaps three 25-note semitone rows running simultaneously. Returning to the octave as the equivalence interval, we have a 36-note scale exhibiting a group property very similar to that of the 25-note scale: $36 = 6^2$, and $25 = 5^2$. [14]

Thus the two systems can coexist within the same composition, even being superposed to produce a kind of bitonality. The apparent character of each is deeply changed as it is heard in the context of the other system. Here we deal then with reversible figure-ground relationships. For example, a 12-tone vocal line sung against a 25-note accompaniment will be variously perceived--the voice sometimes causing us to hear the sixth-tones of the accompaniment in a 36-note scale, at other times the 12 notes themselves being assimilated into the 25-note scale, and at times offering a free choice between these alternative Gestalts.

How are such possibilities to be realized? Direct electronic synthesis on tape provides the most immediate solution. But must the electronic composer create his material virtually anew for each composition? Here we propose a search for new media of such general validity that a concerted effort can be made toward their realization. In the present case we imagine an orchestra (perhaps divided into three differently tuned choirs) to play with, and/or in antiphony to, a fourth-generation electronic musical instrument. Such an instrument would produce all gradations of tone quality between the two limiting ideals, and should mirror with great sensitivity the player's nuances of expression.[15] Thus by degrees one could be led from the familiar dodecaphonic world out into a lucid, weightless, far from dehumanized, tone state where once-familiar sounds re-enter, now transfigured--and as easily return to the harmonic series *terra firma*.

As described, this may seem hardly more than the shift from tonality to atonality and back, or--more plausibly--the passage through quarter-tone domains, as in Boulez' *Le Visage Nuptial*. But even with a firm acoustical basis to serve as a key to its decoding, our attempt could well suffer the fate of a signal from outer space which we fail to distinguish from the flood of thermal noise. That such mathematical domains exist is now in evidence. What potentialities of life they bear will be disclosed only by well-designed and instrumented probes.

From the standpoint of the old, the new remains ever strange; but encompassed within the new, the old is totally reborn.

Fully to restore the human element in musical creation requires a close coupling of the human body and its actions to instruments. Just as we may ask the electronic organ to respond more widely to the performer's touch, so we may require a future synthesizer to realize programs that are executed with the sweep of the artist's brush. The design of a keyboard is really a problem in 'human engineering,' as that branch of space research concerned with the design of instrument panels, control consoles, and sensor-effector apparatus is called. The musician's training is no less arduous than the astronauts's, nor are the suit and capsule less an extension of the body than is a valued Stradivarius. Unused sense
13 modalities, e.g., through coded patterns of weak electronic stimulation of areas of the body, are now being tapped for multiple-channel communication within the the space-suit; but the possibilities of

amplifiying subliminal vocalizations or using them to control additional musical parameters remain almost totally unexplored. And just for lack of an adequate notation, the enormous variety of unamplified vocal sounds is barely hinted at in even the most far-reaching compositional experiments. A cooperative effort by musicians, artists, and scientists of many persuasions will one day greatly extend the range of perceptual spaces that even now partially link the subjective awareness of one being with the inner state of another.

FOOTNOTES

1. For a scale including the seventh and higher harmonics, see Harry Partch, *Genesis of A Music*, Univ. of Wisconsin Press, 1949.

2. See, for example, Paul Hindemith, *The Craft of Musical Composition*, Vol. 1, Associated Music Publishers, New York, pg. 57 ff.

3. Other types of solution have proved possible only with the relaxation of one or more of the five conditions, and are not considered here.

4. See H. S. M. Coxeter, Chapter 11, and Matila Ghkya, 1946. 2nd Ed., 1977.

5. Karlheinz Stockhausen, pg. 16.

6. Precise measurement show, e.g., that a piano is tuned to overtones and not to exact octaves. This slight difference arises from the finite stiffness of strings.

7. Note that while the presence of different modes of vibration in the sound source is a physical fact, Fourier analysis expresses the purely mechanical assertion that any exactly repeated pattern of vibration (periodic function) can be represented as a sum of sinusoids at frequencies f, $2f$, $3f$, ... Conversely, the new tone structure here proposed produces a never-repeating, aperiodic pattern (more accurately, a multiply-periodic pattern unrepresentable by any single-cycle waveform). This is the real meaning of the absence of a 'fundamental' frequency.

8. The fact that this structure resembles the Phrygian mode more closely than the major seems associated with the fact that the analog of the dominant (E-) is below the new 'subdominant' (F-). The inversion symmetry of the generating operation, and the existence of aural subharmonics give nearly equal acoustical footing to all modes of this and other scales of less than 25 notes.

9. Compare Howard Hanson, cited in 'references.'

10. However, non-redundant pentads do exist. For example, the set 5-3-6-4 omits 1 and 2, but exhibits the remaining ten intervals. Under interval multiplication, it transforms to other non-redundant sets omitting other intervals. See *Tone Space II* in *Die Reihe* VIII, 1968.

11. By beginning with a 5 x 5 non-cylic magic square (a Latin square containing only the digits 1 through 5), one can form a row consisting of non-identical pentachords all of this type. Since each of

the pentads is transposable in the above way, subsidiary rows can be derived from any one of the pentachords and its four permuted transpositions.

12. An interval expressed in cents is given by 1200 log $[(f_2/f_1)]$/log[2] If we subdivide phi into a scale of *n* equal steps, the size of a single scale degree measured in cents is 8.330903 cents/*n*. Hindemith and others have pointed out that true intonation is an impossibility theoretically as well as practically, since all but the primary intervals (fourths and fifths) are overdetermined, and the various functions they fulfill are thus simultaneously defined as slightly different numerical ratios. But it seems just conceivable that the details of this 'fine structure' (a term borrowed by analogy from spectroscopy) contribute to the qualitative distinctiveness of each interval. Similar fine structures exist here. While we have listed only one ratio for each interval, other almost equally simple ratios exist, and their differences may be worthy of investigation. The divergences increase for the more remote intervals; the factors in order of remoteness are: 2, 3, $\sqrt{5}$. In any case, what at first appears as a theoretical weakness may turn out to be a functional strength. [This article was also written long before desktop computers and digital synthesizers allowed effortless modulation using large just intonation arrays. Whether this constitutes 'true' intonation is a matter of definition. -- McLaren]

13. It is remarkable that the great majority of these wider combination tones lie very close to scale degrees of the 25-note scale. Again letting ϕ represent the golden section, we may state some of the more prominent relations:

$\phi^2-1 = \phi$; $\phi^3-1 = 2\phi$; $\phi^4-1 = \sqrt{5}\phi^2$; $\phi^6-1 = 4\phi^3$; $\phi^8-1 = 3\sqrt{5}\phi^4$; $\phi^{10}-1 = 11\phi^5$

$\phi^2+1 = \sqrt{5}\phi$; $\phi^3+1 = 2\phi^2$; $\phi^4+1 = 3\phi^2$; $\phi^6+1 = 2\sqrt{5}\phi^3$; $\phi^8+1 = 7\phi^4$; $\phi^{10}+1 = 5\sqrt{5}\phi^5$

14. Even the triple coincidence of 3-4-5's for the major triad has a parallel here. When points five notes apart on the new scale circle are connected, five overlapping pentagons are formed. Now the ratio of a diagonal to an edge of a regular pentagon is again the golden section, adding to the 'solidity' of this graphical representation. It is this occurrence of the golden section that is responsible for its association with the 5-fold symmetry of many living forms (the distribution of petals in a flower, for example). (Ghyka, Chapter 6)

Ghyka points up the recurring fiveness of living organisms in contrast to the frequency of six-fold symmetries of inorganic forms. This distinction is associated with the regular packing of solid crystal structures versus the organic process of 'gnomonic growth' (Sir D'Arcy Wentworth Thompson, *On Growth and Form*, Cambridge University Press: 1917). It is also the presence of pentagonal structures in a liquid that introduces the 'play' essential to its fluidity. Each of the above statements has a clear-cut
13 scientific basis. But the emergence of the recurring 6-fold symmetries from octave-equivalence and equally prevalent 5-fold symmetries from golden-section-equivalence, is another of those confluences of seemingly-fortuitous numerical relationships that adds to the inner consistency and distinctiveness

of the two systems.

15. Since this article was written digital technology has advanced exponentially. The easiest ways of realizing Walter O'Connell's ideas today would be: [1] Use the acoustic compiler Csound on a personal computer to generate phi timbres outside of real time and play them back from hard disk in real time; [2] Create golden-section timbres with a sample-generation program like SoftSynth and load the sounds into a sampler with large amounts of RAM. The K2000 today allows up to 64 megabytes of RAM, approximately 6 minutes of stereo sampling at 44.1 KHz, long enough for nearly 10 seconds per key of monophonic phi timbres. With a multi-track recorder this gives 8 or 16 different phi timbres; [3] Generate the phi timbres in real time with Symbolic Sound's Capybara parallel-processing DSP engine running the Kyma sound-development language under SmallTalk. [Note by McLaren @ Walter O'Connell's request.]

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AFTERWORD

After the above article was written, by 1968 it was recognized that $\sqrt{5}$ should be regarded as a primary generating ratio on a par with 2, instead of being treated as a secondary generator comparable to 3. (The evidence for this lies in the simplicity of the sum and difference tones produced by both 2 and $\sqrt{5}$ in combination with various powers of phi, the similar behaviors for 2^2 and 5 as well their cubes, and in the appearance of the $\sqrt{5}$ among the combination tones themselves.) This upgrading of $\sqrt{5}$ suggests a revision of the previous listing of intervals in order of complexity. However, that list has been let stand in the article because of the relatively poor representation of the $\sqrt{5}$ in the 25-note scale, which remains unique in its ability to mesh with traditional equal temperament. While the 25-note subdivision of phi (with a scale degree of 33.3236 cents) represents 2 with very great accuracy (to within 0.06 of a scale degree), it poorly represents $\sqrt{5}$ as 16.807°, missing by almost 0.2° (6.43¢), and the error is compounded for powers of $\sqrt{5}$.

In contrast, an 18-note subdivision (with a scale degree of 46.2828¢, about 26 per octave) distributes the errors more evenly among the generators, with an error of 0.07 scale degree for 2, 0.10° for $\sqrt{5}$, and 0.09° for 3---all three within 4.6¢. The previous scale derivation from the circle of octaves is now precluded, since the accumulated error after 18 octaves exceeds a full scale degree. However, a method more akin to the derivation of the traditional diatonic scale and its neighboring chromatic tones, now using all three generators in various combinations (as indicated in the table of scale degrees) avoids accumulating more than three or four such errors, which most often partially cancel. This 18-note subdivision would now seem to be the scale of choice for the fullest exploration of phi, except where coordination with traditional tunings is desired.

If we abandon 3 as a generator, it was noted we could use a 9-note scale, whose degrees very nicely represent each of the generators, 2 and $\sqrt{5}$, together with their reciprocals, product, and ratios. Perhaps this was the basis for John Chowning's creating just such a scale for his Stria, written in 1976 at a time when he was pioneering FM synthesis. (Charles Dodge and Thomas A. Jerse, Computer Music, Schirmer Books, 1985: pp 124-127.) When one sine tone is frequency modulated by another, the resulting spectrum consists of the first ('carrier') frequency, plus or minus integer multiples of the second. Here the two frequencies were in the ratio of the golden mean. Algebraic analysis shows that the result would consist of a few
13 powers of phi together with their products with the $\sqrt{5}$, and other more complexly related partials, producing a kind of 'clouded' version of our phi tone, with only a limited range of powers of phi. Here then, combination tones are introduced by the very method of production, and not just the nonlinearity of the ear. The attempt to incorporate the most prominent of these partials could then lead naturally to a choice of a 9-note subdivision of the 'pseudo-octave,' as it was referred to.

For many, FM synthesis (as in the Yamaha DX and TX series, and later synthesizers incorporating FM) will be one of the more accessible means of making a preliminary exploration of the phi tonality. Computer synthesis is ideal; Brian McLaren will add a note in that regard. But note that even if a CD library of phi tones were available, it would not be of much use to present samplers except for notes of short duration. Because of the aperiodicity (mulitply periodic character) of the phi tone, there is no loop point, and the note produced can be no longer than the sample, unless the awkward joins are somehow obscured.

The predicted sense of an octave-like equivalence was fully confirmed by virtually everyone who heard the few preliminary demonstrations by the late Paul Beaver on his Moog synthesizer in 1967. As expected, the irrational ratios are not perceived as discordant, because of the absence of beats between partials. There may be a faint echo of the open-voiced augmented triad, but its 'stretching' (by thirds of a semitone) and the absence of harmonics largely mitigate this. Moreover, the scale degrees of the 25-note scale are clearly discernible, moreso than might be anticipated, because of the purity of the tone structure. (In contrast to the familiar harmonic series, here the partials of any one tone include no other note of the scale.) There was no opportunity to explore the possible qualititative differences between intervals at that time.

A few subsequent experiments using FM synthesis have tended to emphasize the expected sparseness of the overtone structure of the pure phi tone, and to support the inclusion of combination tones formed between higher powers of phi as additional partials to enrich the range of timbres available. (Note that these soon diverge from the partials automatically produced by FM synthesis, which tend to clutter the musical space.) With the addition of such partials, the notion of vocal and instrumental formants may be useful, reinforcing specified frequency ranges with fixed parametric filters. Despite the fact that the tone is spread out over the entire audio spectrum, it may nevertheless be focussed in a specific frequency range.

Subsequent studies have put less emphasis on symmetry, and instead have strengthened some peculiar parallels with traditional tonality. Here both 2 and $\sqrt{5}$, as the primary generators, function as two 'dominants' (their reciprocals being the subdominants). As the secondary generator, 3 serves as a mediant. Thus a four-note 'tonic' chord can be identified, with its combination tones tending to produce transpositions of its original structure. It appears that other parallels to traditional tonality are present, but can only be confirmed by increasing aural familiarity with the material.

The phi tonality is only one of a great variety of possible musical spaces that await exploration. The above material is offered as an example of how a new tone structure may be postulated, and scales appropriate to it then sought out. In the process, many preconceptions have been challenged: the octave is not the only possible equivalence interval; consonance and the absence of beats are not exclusively tied to whole number ratios, and other types of harmonic relations exist; a high tone need not necessarily be masked by a lower one; a single tone may simultaneously be both high and low; a complex aperiodic tone with no representative wave form can nevertheless have an exceptionally well differentiated pitch, unassignable to any single frequency. A fundamental question now opened concerns the conditions under which a structured set of partials will cohere and be perceived as a single tone. Mathematics may help to reveal some such structures.

Here I want to thank Ervin M. Wilson for his interest, encouragement, and insistence upon seeing that this material be published through the opportunity afforded by this journal. The text you are reading would not exist without Brian McLaren's patient deciphering, setting and resetting of the very rough original typescript.

No references are needed for the aspects of phi that have been used here. Its remarkably diverse properties and ubiquity in nature have barely been touched upon. My original source was Matila Ghyka, The Geometry of Art and Life, Sheed and Ward, 1946, which remarkably remains in print in very inexpensive edition (Dover, 1977), as does H.E. Huntley, The Divine Proportion, Dover, 1970, which was published subsequent to most of this work. It was Paul Hindemith, in his Craft of Musical Composition, Vol 1, Schott, 1942 who first alerted me to the importance of combination tones in musical perception and tonal structures. His physical analysis was probably drawn from a source fundamental to everyone interested in scale construction, Hermann Helmholtz, On the Sensations of Tone, Dover, 1954, which fortunately likewise remains available. No mention has been made here of my examination of possible extensions of phi to the rhythmic domain. For a more general background supporting a variety of applications, I can recommend either of the first-mentioned books above, and/or Jay Kappraff, Connections: The Bridge Between Art and Science, McGraw-Hill, 1991. Kappraff includes a summary of E. Lendvai's now unavailable chapter on Bartok's extensive use of both the golden mean and the Fibonacci numbers in his choice of scales, melodic formations, rhythmic structures, and overall architecture.