and J, when $k=0$, let any player faced with a position for which there is no move lose. Then even when $k=0$, (25) and (26) characterize the safe positions for Games H and J respectively. If $k=0$, a position (x, y) is safe in Game I if and only if $x=y$.

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RECURRENT SEQUENCES AND PASCAL'S TRIANGLE

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A. Introduction. The Fibonacci sequence can be found by summing the terms on successive "diagonals" of Pascal's Triangle [1]. J. Raab [2] generalized this procedure to show other sets of parallel diagonals generating different recurrent sequences. This generalization is essentially the same as Phase One in what is to follow. The purpose of this paper is to show that there exist infinitely many more recurrent sequences within Pascal's Triangle by summing the terms on diagonals of different slopes. Each sequence shall be of the type such that each term is the sum of *two* former terms. There is also a unique relationship between just what two terms are involved and the slope of the diagonals being considered.

For this purpose it is convenient to arrange the terms of Pascal's Triangle on the point-lattice determined by the nonnegative integral points of a rectangular coordinate system. (See Figure 1.)

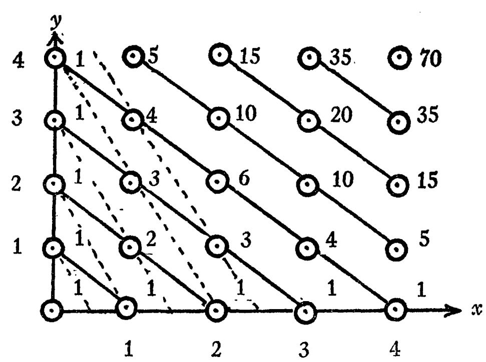

FIG. 1

With this arrangement the coordinates, (x, y), of the lattice point uniquely determine the location and value of a Pascal number $\binom{n}{r}$. The value is seen to be

$$\text{A(1)} \qquad \binom{n}{r} = \frac{(x+y)!}{x!\cdot y!},$$

since $n=x+y$ and $r=y$ (or x, because of the symmetry involved).

B. Phase one. Consider the linear equation

$$\text{B(1)} \qquad x + y = n \qquad \text{for } n = 0, 1, 2, \cdots.$$

This equation represents the nth row (diagonal) of Pascal's Triangle. If we sum the Pascal numbers on each row determined by B(1) for successive values of n, we obtain the sequence

$$\text{B(1.1)} \qquad 1, 2, 4, 8, \cdots, 2^n, \cdots$$

whose recurrence relation is given by

$$\text{B(1.2)} \qquad P_n = P_{n-1} + P_{n-1},$$

where $P_0, P_1, \cdots, P_n, \cdots$ denote the terms of the sequence, and the formula for the nth term is given by

$$\text{B(1.3)} \qquad P_n = 2^n = \sum_{\substack{x=0,y=0\\x+y=n}}^{n,n} \frac{(x+y)!}{x!\cdot y!} = \sum_{r=0}^{n} \binom{n}{r}.$$

(*Note*: the nth term is the term formed by summing all of the Pascal numbers on the line $x+y=n$ and, if we were counting the terms, this term would actually be the $(n+1)$th term in the sequence.)

The sum of the first n terms of the sequence is given by

$$\text{B(1.4)} \qquad \sum_{k=0}^{n-1} P_k = P_n - 1.$$

Now consider the linear equation

$$\text{B(2)} \qquad 2x + y = n \qquad \text{for } n = 0, 1, 2, \cdots.$$

This equation represents the nth diagonal referred to above used to obtain the nth Fibonacci number. By summing the Pascal numbers on each diagonal determined by B(2) for successive values of n (see the dotted lines, Figure 1), we obtain the sequence

$$\text{B(2.1)} \qquad 1, 1, 2, 3, 5, \cdots, F_n, \cdots$$

whose recurrence relation is given by

$$\text{B(2.2)} \qquad F_n = F_{n-2} + F_{n-1}.$$

The formula for the nth term is given by

B(2.3) $$F_n = \sum_{\substack{x=0,\,y=0\\2x+y=n}}^{[n/2],\,n} \frac{(x+y)!}{x!\cdot y!} = \sum_{r=0}^{[n/2]} \binom{n-r}{r}$$

where [ ] denotes the greatest integer function and the sum of the first n terms of the sequence is given by

B(2.4) $$\sum_{k=0}^{n-1} F_k = F_{n+1} - 1.$$

Next consider the linear equation

B(3) $$3x + y = n \qquad \text{for } n = 0, 1, 2, \cdots.$$

In a way similar to that used above we establish the sequence

B(3.1) $$1, 1, 1, 2, 3, 4, 6, 9, \cdots, G_n, \cdots$$

whose recurrence relation is given by

B(3.2) $$G_n = G_{n-3} + G_{n-1}.$$

The formula for the nth term is given by

B(3.3) $$G_n = \sum_{\substack{x=0,\,y=0\\3x+y=n}}^{[n/3],\,n} \frac{(x+y)!}{x!\cdot y!} = \sum_{r=0}^{[n/3]} \binom{n-2r}{r}$$

and the sum of the first n terms of the sequence is given by

B(3.4) $$\sum_{k=0}^{n-1} G_k = G_{n+2} - 1.$$

Now consider the linear equation

B(j) $$jx + y = n \qquad \text{for } n = 0, 1, 2, \cdots \quad \text{and } j = 1, 2, 3, \cdots.$$

This equation, by the procedure referred to above, establishes a sequence whose recurrence relation is given by

B(j.2) $$T_n = T_{n-j} + T_{n-1}.$$

The formula for the nth term is given by

B(j.3) $$T_n = \sum_{\substack{x=0,\,y=0\\jx+y=n}}^{[n/j],\,n} \frac{(x+y)!}{x!\cdot y!} = \sum_{r=0}^{[n/j]} \binom{n-(j-1)r}{r}$$

and the sum of the first n terms of the sequence is given by

B(j.4) $$\sum_{k=0}^{n-1} T_k = T_{n+(j-1)} - 1.$$

For a proof of B(j.2) and B(j.4), see E(a.2) and E(a.4).

C. Phase two. In Phase One each linear equation had the coefficient pair $\langle j, 1\rangle$, giving rise to infinitely many recurrent sequences. We now consider any coefficient pair $\langle j, 2\rangle$, determining the following equations:

C(1) $$x + 2y = n$$

C(2) $$2x + 2y = n$$

C(3) $$3x + 2y = n \qquad \text{for } n = 0, 1, 2, \cdots.$$

$$\vdots$$

C(j) $$jx + 2y = n$$

Several of these are equivalent to cases already discussed, namely,

(i) any coefficient pair $\langle a, b\rangle$ will yield the same sequence and recurrence relation as the pair $\langle b, a\rangle$, because of the symmetry of the Pascal Triangle about the line $y=x$;

(ii) any coefficient pair $\langle a, b\rangle$ presupposes the fact that a and b are relatively prime, since if they are not, then two possibilities occur. Either the equation will be reduced by dividing thru by the greatest common divisor or it will be such that the given value of n will not be divisible by the g.c.d. of a and b. If the equation is reduced, it will have been treated in an earlier phase, and if the equation cannot be reduced, there will be no integral solutions (see [**3**]), and thus no sequence will be determined;

(iii) even when a and b are relatively prime, there will be cases where $ax+by=n$ will not have nonnegative integral solutions. This means that for those particular values of n, the recurrent sequence derived from $ax+by=n$ will have zero as a value for those nth terms in the sequence, since there will be no Pascal numbers to sum. Thus we establish the following useful

LEMMA. *The equation $ax+by=n$, where a, b, and n are nonnegative integers and $ab\neq 0$ and $(a, b)=1$, will not have nonnegative integral solutions when $n=ab-(ja+kb)$, where $j, k=1, 2, 3, \cdots$.*

Proof. Assume $n=ab-(ja+kb)$ so that $ax+by=ab-ja-kb$ with nonnegative solution (x, y). Thus $a(x+j)+b(y+k)=ab$. Let $X=x+j$ and $Y=y+k$; then $aX+bY=ab$. It is important to note here that both X and Y, as well as both a and b, are greater than or equal to one. We can now transform the above equation to $b=X+(bY)/a$ or $b-X=(bY)/a$. Now $b-X$ is an integer; therefore a divides Y since a and b are relatively prime. Suppose $Y/a=r$ so that $Y=ar$.

Similarly we can show that $a-Y=(aX)/b$ and hence conclude that b divides X. Suppose $X/b=s$ so that $X=bs$. Then, by substitution, we have $abs+abr=ab$ or $ab(r+s)=ab$. Therefore $r+s=1$. But both r and s are greater than or equal to one; therefore we have a contradiction.

Thus, in view of the above discussion, the first new case in Phase Two is C(3):

C(3) $$3x + 2y = n \qquad \text{for } n = 0, 1, 2, \cdots.$$

By summing the Pascal numbers on each diagonal determined by C(3) for successive values of n we find that there is no positive integral solution for $n=1$, since, $1=3\cdot 2-(3+2)$ as predicted by the lemma; therefore the sequence is

C(3.1) $$1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, \cdots, H_n, \cdots$$

whose recurrence relation is given by

C(3.2) $$H_n = H_{n-3} + H_{n-2}.$$

The formula for the nth term is given by

C(3.3) $$H_n = \sum_{\substack{x=0,y=0\\3x+2y=n}}^{[n/3],[n/2]} \frac{(x+y)!}{x!\cdot y!}.$$

A formula for H_n in terms of n and r could also be given; however, it actually requires *two* formulas and, in general, the formula will require b different representations, one for each of the different values in the residue class of n (mod b). Phase Three will need three formulas, etc. More will be said about this in the discussion of the general phase.

The formula for the sum of the first n terms of the sequence is given by

C(3.4) $$\sum_{k=0}^{n-1} H_k = H_{n+2} + H_{n+1} - 1.$$

Now by similar considerations of the next new case,

C(5) $$5x + 2y = n \qquad \text{for } n = 0, 1, 2, \cdots,$$

we find that there are no solutions for $n=1$ and $n=3$, since $1=5\cdot 2-(5+2\cdot 2)$ and $3=5\cdot 2-(5+2)$. Hence, by summing the Pascal numbers on each successive diagonal determined by C(5), we obtain the sequence

C(5.1) $$1, 0, 1, 0, 1, 1, 1, 2, 1, 3, 2, 4, 4, 5, 7, 7, 11, 11, \cdots, I_n, \cdots$$

whose recurrence relation is given by

C(5.2) $$I_n = I_{n-5} + I_{n-2}.$$

The formula for the nth term is given by

C(5.3) $$I_n = \sum_{\substack{x=0,y=0\\5x+2y=n}}^{[n/5],[n/2]} \frac{(x+y)!}{x!\cdot y!},$$

and the sum of the first n terms is given by

C(5.4) $$\sum_{k=0}^{n-1} I_k = I_{n+4} + I_{n+3} - 1.$$

Now consider the general case of Phase Two,

C(j) $$jx + 2y = n \qquad \text{for } n = 0, 1, 2, \cdots \qquad \text{and } j = 1, 2, 3, \cdots.$$

This equation establishes a sequence whose recurrence relation is given by

C(j.2) $$T_n = T_{n-j} + T_{n-2}$$

The formula for the nth term is given by

C(j.3) $$T_n = \sum_{\substack{x=0,y=0\\ jx+2y=n}}^{[n/j],[n/2]} \frac{(x+y)!}{x!\cdot y!}$$

and the sum of the first n terms of the sequence is given by

C(j.4) $$\sum_{k=0}^{n-1} T_k = T_{n+(j-1)} + T_{n+(j-2)} - 1.$$

D. Phase three. Consider the equation

D(j) $\qquad jx + 3y = n \quad \text{for} \quad n = 0, 1, 2, \cdots \quad \text{and} \quad j = 1, 2, 3, \cdots$

where the pair $\langle j, 3\rangle$ complies with the remarks made in section C. By summing the Pascal numbers on each diagonal given by D(j) for successive values of n we obtain the sequence whose recurrence relation is given by

D(j.2) $$T_n = T_{n-j} + T_{n-3}.$$

The formula for the nth term is given by

D(j.3) $$T_n = \sum_{\substack{x=0,y=0\\ jx+3y=n}}^{[n/j],[n/3]} \frac{(x+y)!}{x!\cdot y!},$$

and the sum of the first n terms is given by

D(j.4) $$\sum_{k=0}^{n-1} T_k = T_{n+(j-1)} + T_{n+(j-2)} + T_{n+(j-3)} - 1.$$

Proof of the formulas of Phase Two and Phase Three will be covered by the proofs in the general phase that follows.

E. Phase b. In general, the equation

E(a) $\qquad ax + by = n \quad \text{for} \quad n = 0, 1, 2, \cdots \quad \text{and} \quad a, b = 1, 2, 3, \cdots,$

where the pair $\langle a, b\rangle$ complies with the remarks made in section C, will, by summing the Pascal numbers on each diagonal for successive values of n, yield a recurrent sequence whose recurrence relation is given by

E(a.2) $$T_n = T_{n-a} + T_{n-b}.$$

Proof. The first term in the series representing T_n, as defined by E(a.3) below, will be $(x+y)!/(x!\cdot y!)$, where x and y satisfies E(a). In the notation of A(1) this will equal

$$\binom{x+y}{x}.$$

Suppose that this first solution of x and y is the one where x is minimum (and hence y is maximum); then the next solution would be $(x_{\min}+b, y_{\max}-a)$ and the next would be $(x_{\min}+2b, y_{\max}-2a)$, etc., until $y_{\max}-ra$ becomes $y_{\min}$, where r is the greatest integer in the quotient $n/(ab)$. (This is a modified form of a standard result of number theory; see, for example, [3].) Now if we let $k=x_{\min}+y_{\max}$, we can write the first few terms of T_n as follows

$$T_n = \binom{k}{x} + \binom{k+b-a}{x+b} + \binom{k+2b-2a}{x+2b} + \cdots, \tag{1}$$

where the x refers to only $x_{\min}$.

Next we look at T_{n-a}. The first term of this series will be of the form $\binom{k'}{x'}$ where k' and x' are related to k and x above in the following manner. First we note that

$$ax' + by' = n - a, \quad \text{and} \quad x' + y' = k'. \tag{2}$$

Now since a, b and n have all been fixed we find that $x'=x_{\min}-1$ and $y'=y_{\max}$ is a solution, which upon substitution satisfies $ax+by=n$. Furthermore, since y' is the same $y_{\max}$ as found in the consideration of T_n, it will also be the maximum y in the consideration of T_{n-a}, since $n-a$ is less than n; hence x' is the corresponding minimum value. This makes $k'=k-1$. Therefore, we can write the first few terms of T_{n-a} as follows:

$$T_{n-a} = \binom{k-1}{x-1} + \binom{k-1+b-a}{x-1+b} + \binom{k-1+2b-2a}{x-1+2b} + \cdots, \tag{3}$$

where, again, x refers to the original $x_{\min}$. If $x_{\min}$ is zero to begin with, then for the solution of (2) we choose $x'=x_{\min}+b-1$ and $y'=y_{\max}-a$ and this choice modifies (3) only to the extent that the first term is omitted in the series for T_{n-a}.

Next we consider T_{n-b}. The first term of this series will be of a form $\binom{k''}{x''}$ where

$$ax'' + by'' = n - b, \quad \text{and} \quad x'' + y'' = k''. \tag{4}$$

We note here that

$$\binom{k''}{x''} = \binom{k''}{y''}$$

and we find that $x''=x_{\min}$ and $y''=y_{\max}-1$ is the appropriate solution of (4). Therefore, $k''=k-1$ and we can write the first few terms of T_{n-b} as follows:

$$T_{n-b} = \binom{k-1}{x} + \binom{k-1+b-a}{x+b} + \binom{k-1+2b-2a}{x+2b} + \cdots, \tag{5}$$

where again x refers to the original $x_{\min}$. Now if we add the two series (3) and (5) termwise, we observe the result that we desired, namely, series (1), through the use of Pascal's Rule, which determines the very nature of Pascal's Triangle.

The general nth term of this sequence in terms of Pascal numbers is given by

E(a.3)
$$T_n = \sum_{\substack{x=0,y=0 \\ ax+by=n}}^{[n/a],[n/b]} \frac{(x+y)!}{x!\cdot y!};$$

and the sum of the first n terms of this sequence is given by

E(a.4)
$$\sum_{k=0}^{n-1} T_k = T_{n+(a-1)} + T_{n+(a-2)} + \cdots + T_{n+(a-b)} - 1.$$

Proof (by induction). In the development of the phases above a was always greater than or equal to b. The remarks in section C indicate that this is an arbitrary choice because of the symmetry involved; however, one or the other choices must be made, but not both. We will assume here that $a \geqq b$. We divide the proof into two parts.

PART 1. We establish the formula for $n=1$. E(a.4) becomes

$$T_0 = T_a + T_{a-1} + T_{a-2} + \cdots + T_{a-b+1} - 1.$$

Now $T_0=1$, since $ax+by=0$ has only the single solution $(0, 0)$ and $(0+0)!/(0!\cdot 0!) =1$. Also we see that $T_a=1$, since the only solution of $ax+by=a$ is $(1, 0)$ and $(1+0)!/(1!\cdot 0!)=1$. In considering the other terms, $T_{a-1}, T_{a-2}, \cdots, T_{a-b+1}$, we find that the only solutions to the corresponding equations are, in all cases, $x=0$ and y equal to $(a-1)/b$, $(a-2)/b, \cdots, (a-b+1)/b$ respectively. Now only one number of this set of values is integral since the set $a, a-1, a-2, \cdots, a-b+1$ forms a residue class modulo b and, since a and b are relatively prime, the value a is omitted from consideration. All other solutions are nonintegral and therefore discarded and T_k for those values equals zero. Let k' be the one value that yields the integral solution and let k'' be that solution. Then

$$T_k = \frac{(0+k'')!}{0!\cdot k''!} = 1$$

Hence $T_0 = T_a + T_{k'} - 1$ or $1 = 1+1-1$ an identity.

PART 2. We assume the formula is true for n and show that then it is also true for $n+1$. To both sides of E(a.4) add T_n. Thus

$$\sum_{k=0}^{n} T_k = T_{n+(a-1)} + T_{n+(a-2)} + \cdots + T_{n+(a-b)} - 1 + T_n.$$

But from E(a.4) we have

$$\sum_{k=0}^{n} T_k = T_{n+1+(a-1)} + T_{n+1+(a-2)} + \cdots + T_{n+1+(a-b)} - 1.$$

We must show that the right members of the above equations are equal. Upon equating these two members and simplifying we have $T_{n+a-b} + T_n = T_{n+a}$. But we know from E(a.2) that $T_k = T_{k-a} + T_{k-b}$. Thus if $k = n+a$, we have the exact statement above.

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AN EXPLICIT EXPRESSION FOR BINARY DIGITAL SUMS

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1. Introduction and statement of results. If $\alpha(\kappa, r)$ denotes the sum of the digits of κ when κ is represented in base r, then it has been established ([**1**], [**2**], [**3**]) that

$$A(n, r) \equiv \sum_{\kappa<n} \alpha(\kappa, r) = \tfrac{1}{2}(r-1)n \log_r n - E(n, r) \tag{1.1}$$

where $E(n, r) = O(n)$. The purpose of this paper is to present a more detailed examination of $E(n, r)$ for the special case in which r is two. An interesting consequence of the investigation is that $E(n, 2)$ is expressed in terms of a continuous nondifferentiable function similar to that given by van der Waerden [**4**]. The function may be defined as follows: Let $g(x)$ be periodic of period one and defined on $[0, 1]$ by

$$g(x) = \begin{cases} \tfrac{1}{2}x, & 0 \leqq x \leqq \tfrac{1}{2} \\ \tfrac{1}{2}(1-x), & \tfrac{1}{2} < x \leqq 1. \end{cases} \tag{1.2}$$

The function

$$f(x) \equiv \sum_{i=0}^{\infty} \frac{1}{2^i} g(2^i x) \tag{1.3}$$

can be shown to be nondifferentiable. The relation between this function and $E(n, 2)$ is demonstrated in the following theorems:

THEOREM 1. *If the integer n is written $n = 2^m(1+x)$, $0 \leqq x < 1$, then*

$$E(n, 2) = 2^{m-1}\{2f(x) + (1+x)\log_2(1+x) - 2x\}.$$

THEOREM 2. *If n is represented as in Theorem* 1, *then*

$$E(n, 2) < 2^{m-1}\{5/3 \log_2 (5/3) - 2/3\}$$

and the constant cannot be reduced.

Drazin and Griffith [**2**] and more recently Clements and Lindstrom [**5**] have shown that

$$E(n, 2) < 2^{m-1}(1+x)\log_2(4/3) \tag{1.2}$$